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Solution of Dirichlet Pure Diffusion Problem Using Galerkin Method

Research Article

Goh Rui Zher*

Department of Mechanical Engineering, Faculty of Engineering, UCSI University, Kuala Lumpur, Malaysia

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ABSTRACT

Dirichlet pure diffusion problem refers to physical problem which only includes diffusion effect on the transport of variable within a continuum. It can be solved by central differencing finite volume method with good accuracy, provided that sufficient grids are prescribed. However, the solution using finite volume method may have high computational cost when the large number of grids takes place. Hence, in this study, Galerkin method is used as an alternative to solve the Dirichlet pure diffusion problem. Galerkin method is a numerical approach used to obtain numerical approximation by converting a continuous operator problem such as differential equation to a discrete problem. However Galerkin method is proven not to be a good tool for solution of this higher order partial differential equation physical problems due to its low accuracy, high complexity and unsatisfactory accessibility.

1. Introduction

The heat diffusion is a process where the thermal energy exchanged between two physical systems. The rate of heat transfer between two media is often depending on the temperature difference as well as the thermal conductivity of the media [1,2]. Thermal conductivity is defined as the property of material to conduct heat where by the higher the thermal conductivity of a material, the higher the thermal energy being transfer though a material [3].

There are various heat engineering problems that require engineers to solve using numerical approximation [4], while one of the numerical method that widely used is Galerkin method [5]. Galerkin method may outperform the ordinary numerical schemes in terms of computational efficiency, since the grids and discretization is not required. However, the performance and numerical accuracy of Galerkin method compared with the conventional numerical schemes such as finite volume methods, are not well studied.

In this paper, Dirichlet heat diffusion problem on a two-dimensional plate is taken as the case study to examine the numerical performance of Galerkin method. With this numerical approach available for the application of adiabatic heat diffusion problems, the behaviour of heat transferred in a

**E-mail address: ruizherg@gmail.com*

medium can be estimated. In the work, heat generation and work done in the system is ignored, while there is no pressure gradient involved.

2. Methodology

The scopes of study for this paper are two dimensional heat dissipation, no heat convection and negligible heat generation. Hence, the energy equation is simplified to:

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}=0 \quad (1)$$

The energy equation obtained will be used to obtain the Residual of the Galerkin Method by substituting the differential equation into the energy equation.

Consider a two dimensional plate with specify boundary condition as shown in **Fig. 1.**

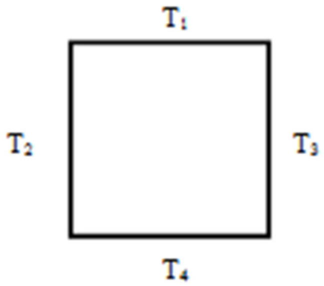

Figure 1. Two dimensional plate with prescribed temperature

where, T_1 = 25°C; T_2 = 50°C; T_3 = 75°C and T_4 = 100°C. A trial function is important in the application of Galerkin method. For trial function 1

$$T = a_1\left(x^2y^2\text{-}xy^2\right) + a_2(x^2y\text{+}xy) + a_3\left(x\text{-}x^2\right) + 80y^2 - 110y + 20x + 85 \quad (2)$$

Eq. (3) and (4) is obtained by apply second order differentiation to the trial function

$$\frac{\partial^2 T}{\partial x^2} = 2a_1y^2\text{+}2a_2y\text{-}2a_3 \quad (3)$$
$$\frac{\partial^2 T}{\partial y^2} = 2a_1x^2\text{-}2a_1x\text{+}160 \quad (4)$$

By substituting Eq. (3) and (4) into Eq. (2), the residual of Galerkin method (R) is obtained.

$$R = 2a_1y^2\text{+}2a_2y\text{-}2a_3\text{+}2a_1x^2\text{-}2a_1x\text{+}160 = 0 \quad (5)$$

In Galerkin Method, the weighting function is equal to the trial function. Therefore, for weighing function for Domain 1, 2 and 3 is in Eq (6) to (8) respectively.

$$w_1 = a_1(x^2y^2 - xy^2) \quad (6)$$
$$w_2 = x^2y + xy \quad (7)$$
$$w_3 = x - x^2 \quad (8)$$

The Galerkin equations formed will be:

$$\int_{y2}^{y1}\int_{x2}^{x1} w_1 \mathrm{R}\ dx\, dy = 0$$
$$\int_{y2}^{y1}\int_{x2}^{x3} w_2 \mathrm{R}\ dx\, dy = 0$$
$$\int_{y2}^{y1}\int_{x3}^{x4} w_3 \mathrm{R}\ dx\, dy = 0$$

Therefore,

$$\int_{100}^{25}\int_{50}^{58.33} (x^2y^2\text{-}xy^2)(\ 2a_1y^2+2a_2y\text{-}2a_3+2a_1x^2\text{-}2a_1x+160)\ dx\, dy = 0 \quad (9)$$
$$\int_{100}^{25}\int_{58.33}^{66.67} (x^2y + xy)(2a_1y^2+2a_2y\text{-}2a_3+2a_1x^2\text{-}2a_1x+160)\ dx\, dy = 0 \quad (10)$$
$$\int_{100}^{25}\int_{66.67}^{75} (x - x^2)(2a_1y^2+2a_2y\text{-}2a_3+2a_1x^2\text{-}2a_1x+160)\ dx\, dy = 0 \quad (11)$$

The parameter of Galerkin method can be obtained by solving Eq. (9) to (11) simultaneously. The values of a_1, a_2 and a_3 are -4.9637×10^{-3}, -0.5089 and -1.6528 respectively. Therefore the Galerkin equation is obtained:

$$T = -4.9637\times10^{-3}(x^2y^2\text{-}xy^2)\text{-}0.5089(x^2y + xy)\text{-}1.6528(\ x - x^2) + 80\ y^2\ \text{-}110y + 20x + 85 \quad (12)$$

3. Numerical Results

The result obtained from the explicit method is used to compute the errors and the accuracy of the Galerkin heat equations obtained. **Fig 2** is the temperature distribution on the two dimensional plate via finite volume explicit method.

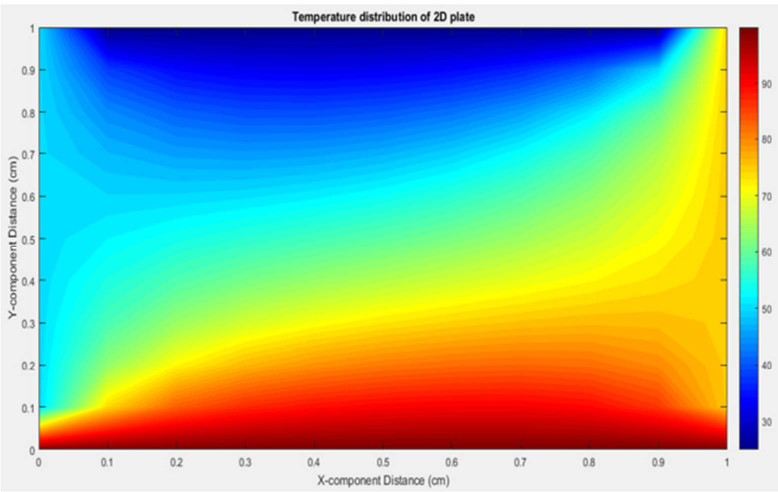

Fig. 2. Contour plot of temperature distribution via explicit method

The equation obtained using Galerkin method is generated on the mesh using Matlab and the result for the temperature distribution of each node of the mesh and the contour plot of the heat diffusion is obtained. **Fig. 3** illustrates the temperature on each node of the mesh and the contour plot of this Galerkin Heat Equation.

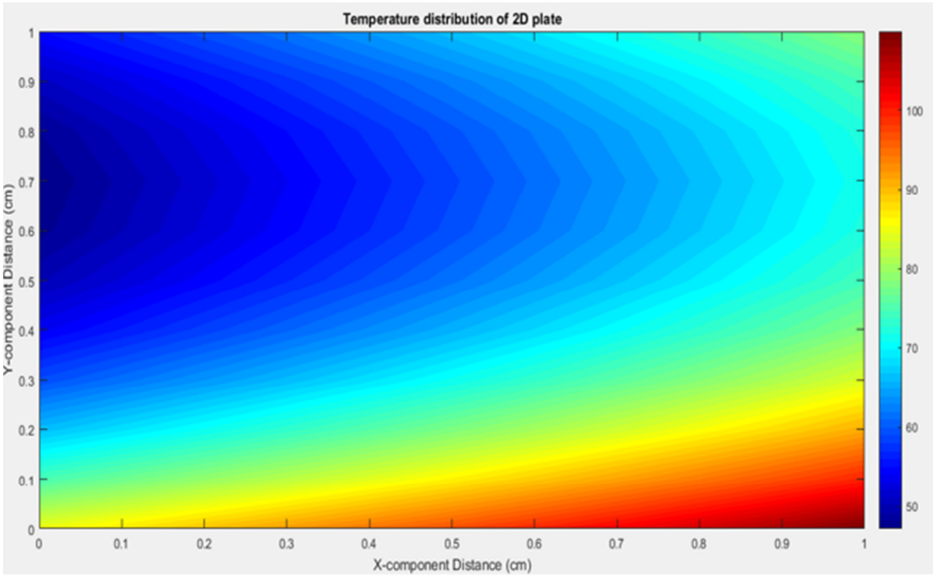

Fig. 3. Contour plot of temperature distribution via Galerkin Method

The accuracy of Galerkin method can be determined by comparing the temperature obtained from both numerical methods. **Fig. 4** is the temperature difference plot between the temperature field obtained via finite volume method and Galerkin method.

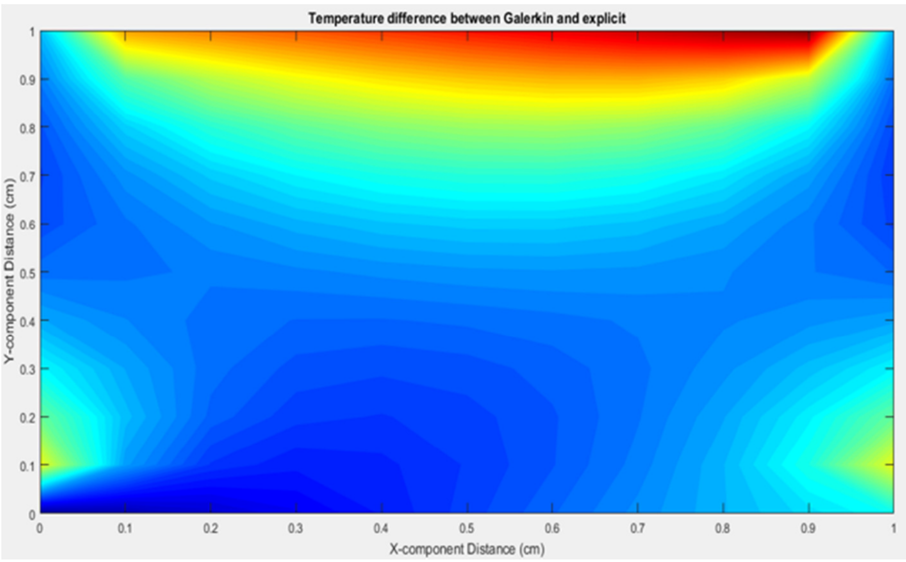

Fig. 4. Temperature difference between Galerkin and finite volume explicit method

The percentage of error for each boundary condition is computed by finding the percentage of difference of the average value of each boundary condition of explicit method and Galerkin method. The percentage of error can be obtained by:

$$\%\ \text{error} = \frac{\text{Average T of Explicit Method-Average T of Galerkin Method}}{\text{Average T of Explicit Method}} \times 100\%$$

The error is computed in **Table 1**.

Table 1: Percentage of error

Boundary Condition	Average Temperature (°C)		Percentage of error (%)
	Explicit method	Galerkin	
Top	25	66.56	-166.24
Left	50	55.30	-10.60
Right	75	79.66	-6.21
Bottom	100	97.13	2.87

From this study, Galerkin method is not a preferable solution for Dirichlet heat diffusion problem. From the result obtained, the error for Galerkin heat diffusion equation is 10% when compared to explicit method and 7% when compared to moving least square method. However, there is large error occur at the top boundary of the mesh for the Galerkin heat diffusion equation when compared to other numerical method. The errors occur is due to the limitation of the parameters of the trial function as each parameter has to multiply with 2 or more unknown which reduce the accuracy of the heat diffusion equation.

4. Limitations of Galerkin Method

From the study, the limitation of Galerkin method is obvious. Galerkin is only able to be applied to solve very simple partial differential equations. Three weaknesses of Galerkin method are discussed in this section.

4.1 *Low accuracy*

Galerkin method has low accuracy compared to other numerical method such as moving least square and explicit method. There is a study conducted by Liu [6] on the accuracy, convergence and stability of Galerkin method. In this study, the author investigates the accuracy, convergence, stability and effectiveness of Galerkin and strong form collocation meshfree methods, with focus on Reproducing Kernel Particle Method (RKPM), Radial Basis Collocation Method (RBCM), and Reproducing Kernel Collocation Method (RKCM). According to the author, Galerkin method yield poor accuracy for the higher order basis and hence the author enhance the accuracy of the Galerkin method by using collocation method and it is found that Galerkin with Radial Basis Collocation Method (RBCM) provide the highest accuracy compared to Reproducing Kernel Particle Method (RKPM) and Reproducing Kernel Collocation Method (RKCM).

4.2 *Complexity*

For higher order application, the trial and test function of the Galerkin method that is able to satisfy the prescribed boundary condition is hard to obtain due to its complexity in satisfy the boundary condition. Therefore, in order to reduce the error of the trial and test function, other numerical method can be applied to obtain the trial function. Moreover, the number of parameters of Galerkin method will increase for higher order application which increases the complexity of the calculation involve in Galerkin method.

4.3 *Accessibility*

Due to the limitations of the parameter and the trial function of Galerkin method, the heat equation obtained in this study is only applicable for the prescribed boundary condition. This is because the parameter of the Galerkin method will change affected by the boundary condition temperature. Hence, the user is required to obtain a new trial function that is able to satisfy the prescribed boundary condition and recalculate the parameter in order to solve the heat diffusion problem.

5. Conclusion

From this study, it is found that Galerkin heat diffusion equation is able to satisfy the prescribed boundary condition with error of 10% except for the top boundary condition. Galerkin method is not a good choice to solve diffusion problem and other physical problems which is complicated. With the limitations of Galerkin method, it is suggested that alternative numerical method such as explicit method and element free Galerkin method to be use to solve the two dimensional heat diffusion problems. In order to minimize the error occur in Galerkin method, other numerical method such as moving least square method should be implemented along with Galerkin method in order to obtain the trial function for Galerkin method.

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