Penerbit Akademia Baru

CFD Letters

Journal homepage: www.akademiabaru.com/cfdl.html
ISSN: 2180-1363

Investigation on Generalised Trapezoidal Differencing Time-Marching Scheme for Modelling of Acoustical Wave

Open Access

Priscilla Francesca[1], Wah Yen Tey[1,2*], Lit Ken Tan[2,3], M.W. Muhieldeen[1]

[1] Department of Mechanical Engineering, Faculty of Engineering, UCSI University, Kuala Lumpur, Malaysia
[2] Takasago i-Kohza, Malaysia-Japan International Institute of Technology, Universiti Teknologi Malaysia, Kuala Lumpur, Malaysia
[3] International Center, Tokyo City University, 1-28-1, Tamazutsumi, Setagaya-ku, Tokyo, 158-8557, Japan

ARTICLE INFO

Article history:
Received 21 December 2019
Received in revised form 17 February 2020
Accepted 22 February 2020
Available online 28 February 2020

ABSTRACT

Generalised Trapezoidal Differencing Time-Marching (GTDTM) Scheme has been developed in this paper to solve non-linear wave equation for acoustical modelling. The method is indeed a combination between Euler explicit method and Euler implicit method. To investigate the computational accuracy for GTDTM, the explicit factor is reduced from 1 (fully explicit) to 0 (fully implicit). The results show that the lower the explicit factor, the propagated wave will be damped with a faster speed. As a remedy, the Courant number is reduced when the explicit factor is small. In fact, Euler implicit scheme is not an effective approach to solve the hyperbolic wave equation. However, the minimum Courant number required due to different explicit factor has been computed too for future implementation of GTDTM.

Keywords:
Wave equation; Explicit time-marching; Implicit time-marching; First order time-marching; Acoustic wave

1. Introduction

Wave equation has been widely applied in engineering physics such as the investigation on tidal wave [1], modelling on ultrasonic pretreatment [2], vibration of quantum particles [3], simulation of mountain wave [4] and formation of Keller-Miksis equation which governs the high-speed bubble dynamics [5,6]. Since wave equation appears as hyperbolic partial differential equation, designing suitable numerical stencils for time-marching scheme is a critical issue among numerical scientists to ensure computational accuracy.

The most popular method in time discretisation of wave equation is Euler explicit method, in which the spatial derivative of the equation is presumed to hold current time step. This is probably due to its simplicity and excellent accuracy, provided that the grid size is small enough. The improvised versions of explicit method comprise Lax method [7-8], Delfim-Soares explicit method [9] and higher order explicit time-stepping method [10]. They are proven to be efficient in the computation of wave equation and other hyperbolic equations.

* *Corresponding author.*
E-mail address: teywy@ucsiuniversity.edu.my (Wah Yen Tey)

Nevertheless, the application of implicit time-stepping method in solving hyperbolic equation is not as favored as explicit scheme. Besides than its complexity, implicit scheme may lead to asymptotically obliterating amplitude [8,11]. To solve the problem, there are basically three types of approaches have been proposed. First method is through combination of the explicit and implicit scheme can be combined, and the example of this is trapezoidal differencing method [8]. Second approach is via the increment of higher order accuracy of Taylor's series expansion, and the examples for this comprise leap-frog method [8], dissipation-adaptive method [11], Lax-Wendroff method [12], high order Runge-Kutta method, and Malkoti method [13]. The third way is via deploying of correction step. The methods which were developed using this method are MacComack method, Rusanov method and Warming-Kutler-Lomax method, and the details of these methods can be found in the work of Tannehill *et al.,* [8]. Implicit-explicit multistep method is recently proposed by Gao and Mei [14] too, yet their numerical scheme, finite element Galerkin approach is applied instead of finite difference method.

However, for most of these improvised methods, their computational performance is left unexamined via practical numerical modelling of wave equation. In this paper, a generalized trapezoidal differencing method [8] will be developed to solve acoustic wave, inspired by the generalization of discretized heat equation from Reith *et al.,* [15]. The possible way to set right the annihilation of wave will be investigated as well.

2. Formulation of Wave Equation

Consider a string as illustrated in Figure 1, the equilibrium of forces in one dimensional domain can be formed from the Newton's second law:

$$F = ma = \rho V a$$

$$\rho\sqrt{\Delta x^2 + \Delta P^2}\,\frac{\partial^2 P}{\partial t^2} = F\left(x+\Delta x,t\right)\sin\left[\theta\left(x+\Delta x,t\right)\right] - F\left(x,t\right)\sin\left[\theta\left(x,t\right)\right] + F_e \tag{1}$$

where ρ, ΔP, F and F_e is the density of string, vertical displacement of the string, string tension and external force applied on the string respectively. By taking Δx to be infinitely close to zero, Eq. (1) can be simplified as in Eq. (2). Then Eq. (2) is differentiated with first derivative of x to form Eq. (3).

$$\rho\sqrt{\Delta x^2 + \Delta P^2}\,\frac{\partial^2 P}{\partial t^2} = F\left(x,t\right)\cos\left[\theta\left(x,t\right)\right] + F_e \tag{2}$$

$$\rho\left[\sqrt{1+\left(\frac{\partial P}{\partial x}\right)^2}\right]\frac{\partial^2 P}{\partial t^2} = \cos\left[\theta\left(x,t\right)\right]\frac{\partial}{\partial x}\left[F\left(x,t\right)\right] + F\left(x,t\right)\frac{\partial}{\partial x}\left\{\cos\left[\theta\left(x,t\right)\right]\right\} + \frac{\partial F_e}{\partial x} \tag{3}$$

Now, by applying the trigonometric equation as in Eqs. (4) and (5) into Eq. (3), the wave equation as in Eq. (6). The physical assumption made during the simplification are: (a) the displacement gradient $\partial P/\partial x$ is equivalent to 1; and (b) in vertical harmonic oscillation, the rate of change of force per unit distance across x-axis will be zero.

$$\cos\theta = \frac{1}{\sqrt{1^2 + \left(\frac{\partial P}{\partial x}\right)^2}} \approx 1 \tag{4}$$

$$\sin\theta = \frac{\frac{\partial P}{\partial x}}{\sqrt{1^2 + \left(\frac{\partial P}{\partial x}\right)^2}} \approx \frac{\partial P}{\partial x} \tag{5}$$

$$\rho\frac{\partial^2 P}{\partial t^2} = F\frac{\partial^2 P}{\partial x^2} \rightarrow \therefore \frac{\partial^2 P}{\partial t^2} = c^2\frac{\partial^2 P}{\partial x^2}\bigg|_{c=\sqrt{\frac{F}{\rho}}} \tag{6}$$

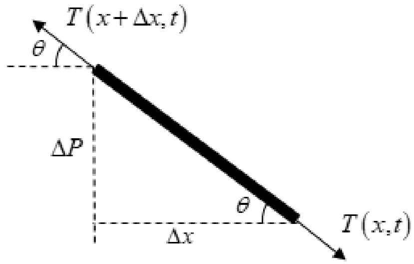


Fig. 1. The propagation of energy via oscillating string

In Eq. (6), c represents the wave speed, which can be further related with the wave frequency f and wavelength λ as shown in Eq. (7). Wave speed for sound is dependent on the transmitting medium and the ambient temperature. For example, the wave of sound moves with the approximate speed of 344 m/s and 1500 m/s in air and water respectively when the ambient temperature is 20°C. The details of the speed of sound can be referred in the work of Müller and Möser [16].

$$c = f\lambda \tag{7}$$

3. Generalised Trapezoidal Differencing Time-Marching Scheme

The discretized equation of Eq. (5) using trapezoidal differencing time-marching scheme [8] is shown in Eq. (8).

$$\frac{P_i^{n+1} - 2P_i^n + P_i^{n-1}}{\Delta t^2} = c^2\frac{1}{2}\left(\frac{P_{i+1}{}^{n+1} - 2P_i^{n+1} + P_{i-1}{}^{n+1}}{\Delta x^2} + \frac{P_{i+1}{}^{n} - 2P_i^{n} + P_{i-1}{}^{n}}{\Delta x^2}\right) \tag{8}$$

Note that for the spatial discretisation, half of the stencils are built based on the future time step, while the other half are holding the current time step. For a generalized equation, Eq. (8) can be further modified to form Eq. (9). Rearrangement of the stencil in Eq. (9) will form Eq. (10).

$$\frac{P_i^{n+1}-2P_i^n+P_i^{n-1}}{\Delta t^2}=c^2\left\{(1-\beta)\frac{P_{i+1}{}^{n+1}-2P_i^{n+1}+P_{i-1}{}^{n+1}}{\Delta x^2}+\beta\frac{P_{i+1}{}^{n}-2P_i^{n}+P_{i-1}{}^{n}}{\Delta x^2}\right\} \tag{9}$$

$$\begin{aligned}&-\alpha(1-\beta)P_{i+1}{}^{n+1}+\left[1+2\alpha(1-\beta)\right]P_i^{n+1}-\alpha(1-\beta)P_{i+1}{}^{n+1}\\&\quad=\alpha\beta P_{i+1}{}^{n}-2(1-\alpha\beta)P_i^n+\alpha\beta(1-\beta)P_{i-1}{}^{n}\end{aligned} \tag{10}$$

where, $\alpha=c^2\dfrac{\Delta t^2}{\Delta x^2}$

In Eq. (8), β (≤ 1) is known as explicit factor, i.e. when β is 0, 1 and 0.5, then Eq. (10) will become fully implicit, fully explicit and trapezoidal differencing method respectively. Eq. (10) can be further re-expressed to yield a matrix as shown in Eq. (11) for the ease of computation.

$$\begin{aligned}&\begin{pmatrix}1+2\alpha(1-\beta) & \alpha(\beta-1) & \cdots & 0\\ \alpha(\beta-1) & 1+2\alpha(1-\beta) & \ddots & \vdots\\ \vdots & \ddots & \ddots & \alpha(\beta-1)\\ 0 & \cdots & \alpha(\beta-1) & 1+2\alpha(1-\beta)\end{pmatrix}\begin{pmatrix}P_1\\P_2\\ \vdots\\P_m\end{pmatrix}^{n+1}\\&\quad=-\begin{pmatrix}P_1\\P_2\\ \vdots\\P_m\end{pmatrix}^{n-1}+\begin{pmatrix}\chi_1\\0\\ \vdots\\ \chi_2\end{pmatrix}++\begin{pmatrix}2(1-\alpha\beta) & \alpha\beta & \cdots & 0\\ \alpha\beta & 2(1-\alpha\beta) & \ddots & \vdots\\ \vdots & \ddots & \ddots & \alpha(\beta-1)\\ 0 & \cdots & \alpha\beta & 2(1-\alpha\beta)\end{pmatrix}\begin{pmatrix}P_1\\P_2\\ \vdots\\P_m\end{pmatrix}^{n}\end{aligned} \tag{11}$$

where:

$$\chi_1=\alpha(1-\beta)S+\alpha\beta S$$
$$\chi_2=\alpha(1-\beta)P_{m-1}{}^{n-3}+\alpha\beta P_{m-1}{}^{n-4}$$

The solution of wave equation via Eq. (11) is named as Generalised Trapezoidal Differencing Time-Marching (GTDTM) scheme. The second term of the right-hand side of Eq. (11) is matrix which imposes the boundary condition. The first term and last term in the matrix represents the source of acoustic wave (Dirichlet boundary condition) and the end-wall boundary condition of the problem domain (Neumann boundary condition). In our problem, the Neumann boundary condition set will enforce a non-reflecting wall for wave. The symbol *m* represents the maximum nodes available in the problem domain, while the mathematical expression of *S* and *α* is shown in Eq. (12) and Eq. (13) respectively,

$$S=P_{\max}\sin(2\pi ft) \tag{12}$$

$$\alpha=\left(c\frac{\Delta t}{\Delta x}\right)^2 \tag{13}$$

where f and t is the wave frequency and instantaneous time of the wave propagation. The frequency and P_{max} set the current work is 20 kHz and 2×10^5 μPa respectively, while speed of sound c is set as 344 m/s. The grid number is 1400. To ensure time marching stability, the Courant-Friedrichs-Lewy condition must be fulfilled [17]. In our simulation the initial Courant number Co applied is 0.5, and its can be expressed mathematically as in Eq. (14). Upon computation of the results, the possible damped pressure will be translated as attenuation of sound pressure level L_p, which can be mathematically represented as in Eq. (15).

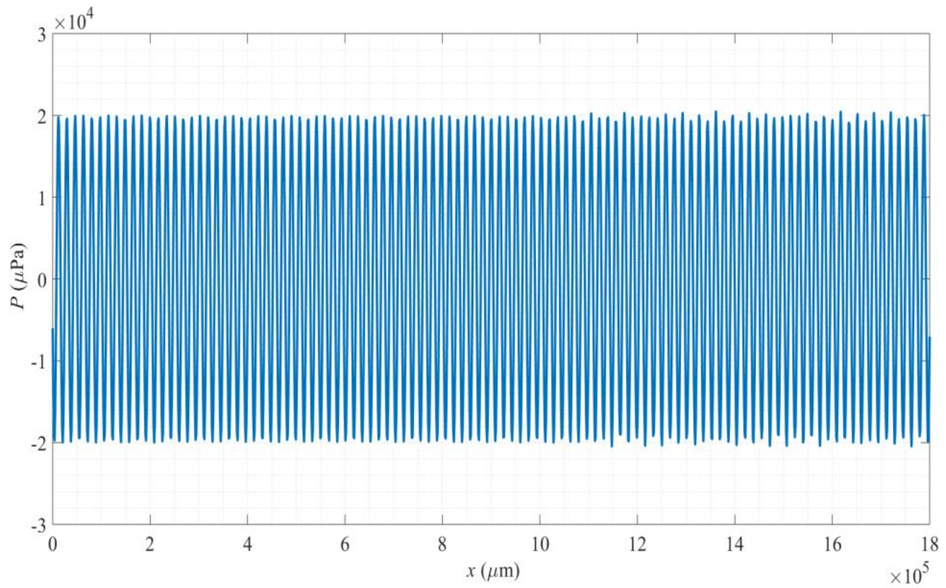
$$Co = \sqrt{\alpha} = c\frac{\Delta t}{\Delta x} \tag{14}$$

$$L_P = 20\log\left(\frac{P^n}{P_{ref}}\right)\Bigg|_{P_{ref}=20\mu Pa} \tag{15}$$

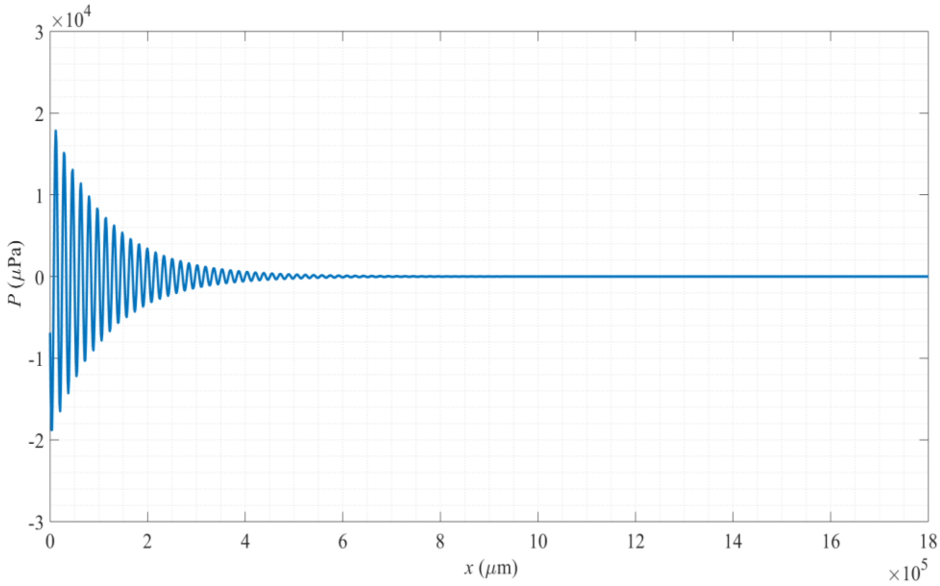
4. Results and Discussion

It can be found that with the increasing value of explicit factor, the damping of the sinusoidal wave can be apparently observed, as shown in Figure 2. At β = 1, which lead to fully explicit scheme, shows an oscillation of wave without "damping" effect, while at β = 0, the solution is fully implicit and the wave propagation will be silenced soon after it travels away from the source. This implies that the increment of weightage for the component of implicit time-marching in Generalised Trapezoidal Differencing Time-Marching Scheme will greatly deteriorate the results. This is in accordance with the findings by Tannehill *et al.,* [8] and Bathe and Baig [18] that the implicit scheme is the source if numerical instability during the non-linear solution of time.

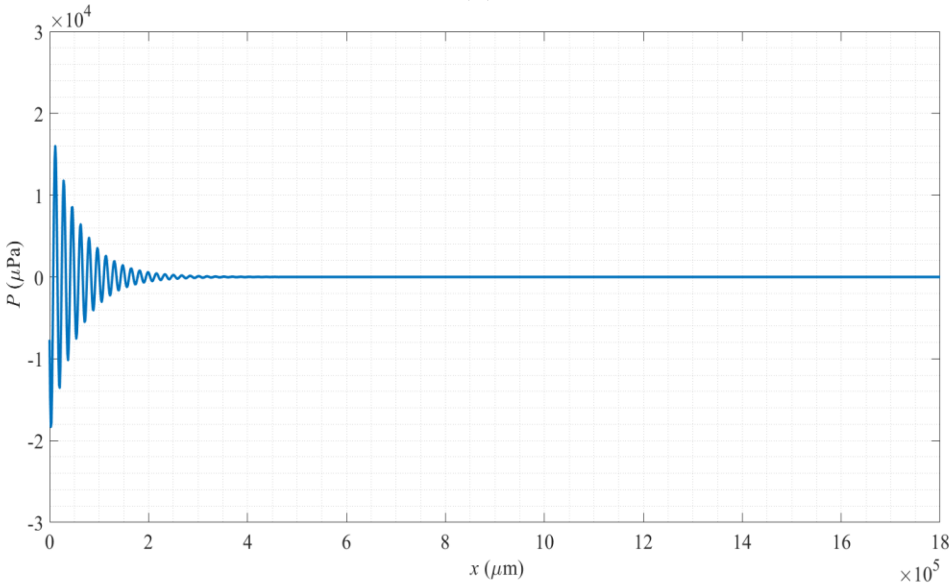
However, the change of β will not affect the wavelength, which is maintained as 0.0172 µm all the time despite the trimming of wave amplitude.



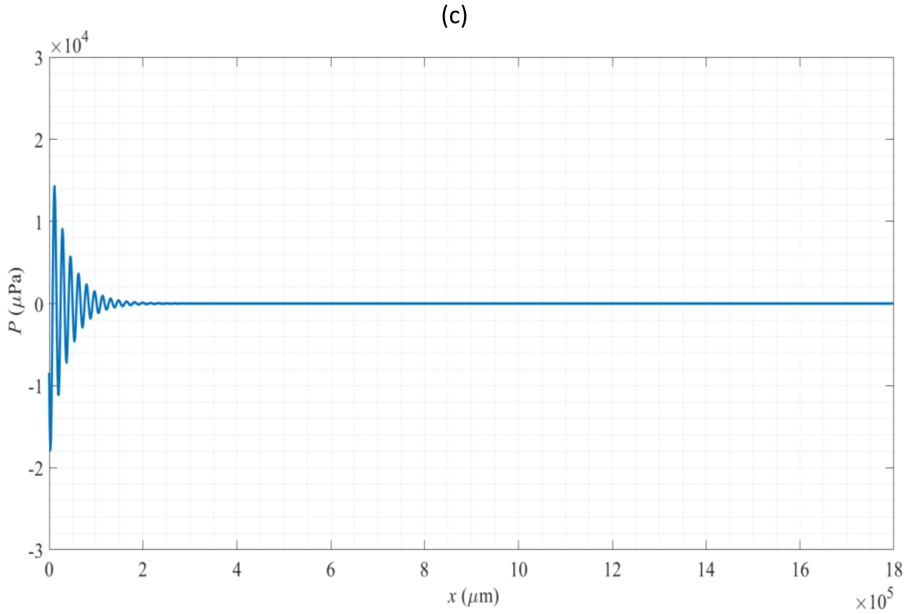
(a)


×10^4
P (μPa)
x (μm)
×10^5
3
2
1
0
-1
-2
-3
0
2
4
6
8
10
12
14
16
18

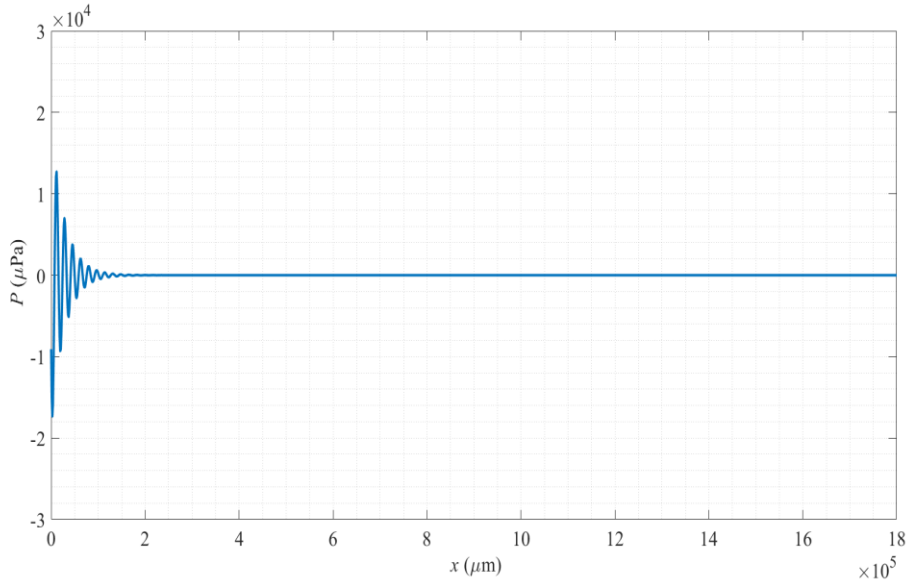
(b)


×10^4
P (μPa)
x (μm)
×10^5
3
2
1
0
-1
-2
-3
0
2
4
6
8
10
12
14
16
18

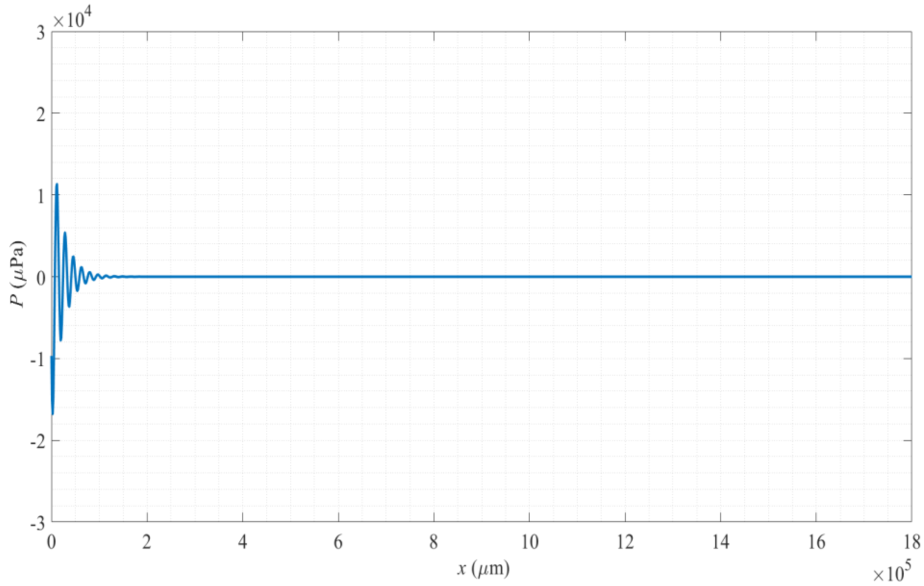
(c)


×10^4
P (μPa)
x (μm)
×10^5
3
2
1
0
-1
-2
-3
0
2
4
6
8
10
12
14
16
18

(d)



(e)



(f)

Fig. 2. Wave propagation when the explicit factor β is: (a) 1.0; (b) 0.8; (c) 0.6; (d) 0.4; (e) 0.2; (f) 0.0 when *Co* = 0.5

The attenuation of wave can be further quantified via the records on the maximum amplitude for every cycle of wave and the resultant sound pressure level reduction, as shown in Figure 3(a) and 3(b) respectively. The maximum amplitude and resultant sound pressure level is reducing in exponential way and negative linear way respectively. Although the blue line as in Figure 3(a) which represents the maximum pressure for β = 1 appears to be unstable, yet there is no annihilation observed. The see-sawing of the results is possibly caused by the insufficient spatial grids.

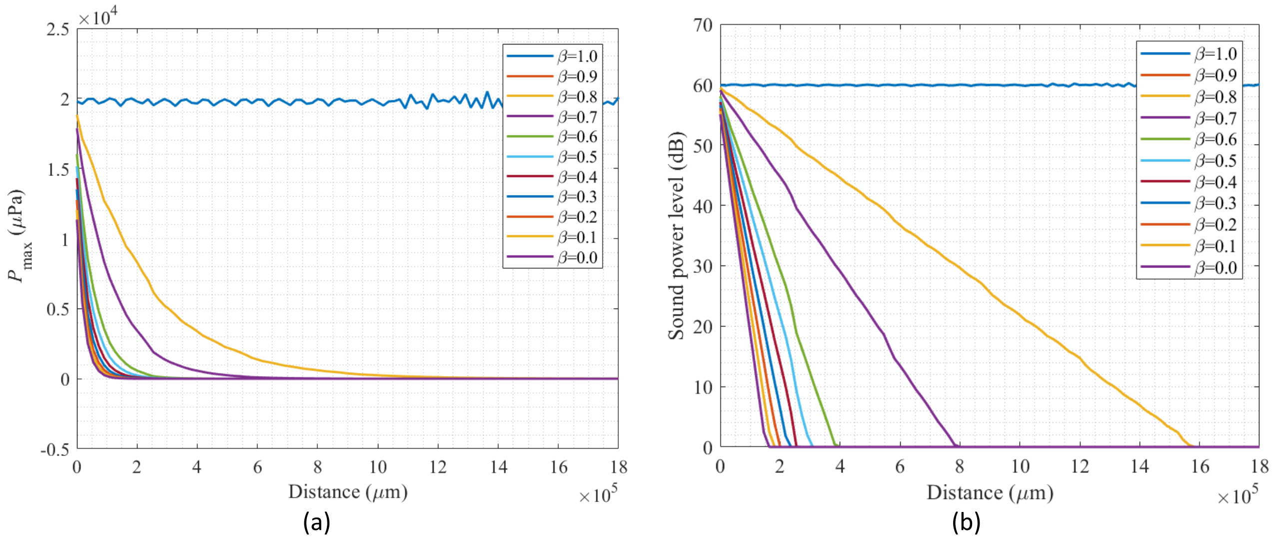


Fig. 3. Attenuation of wave propagation in terms of (a) maximum pressure per cycle; and (b) sound pressure level

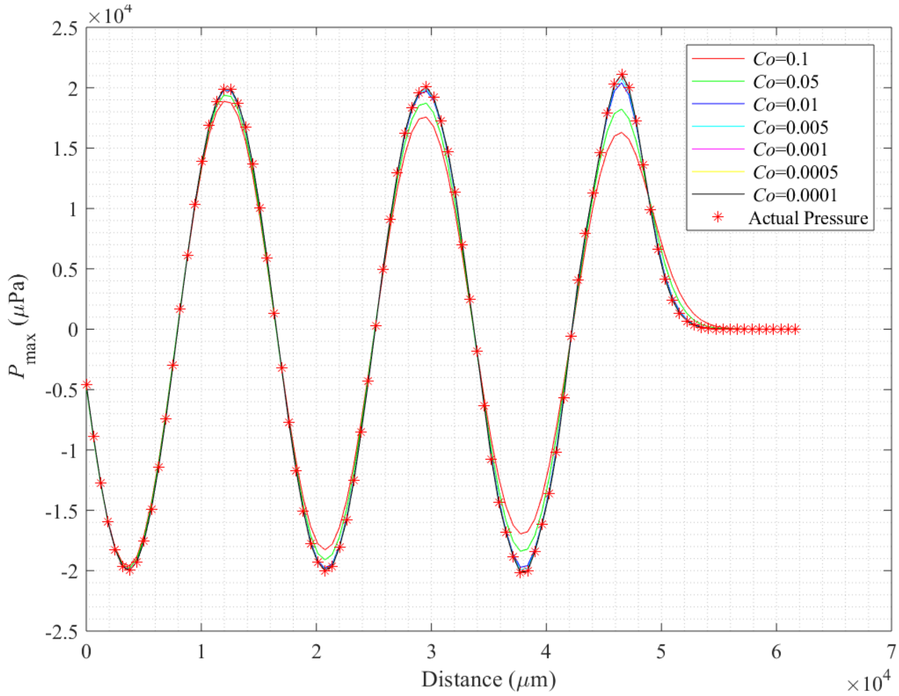
To improve the GTDTM scheme, the remedy is endeavored by reducing the Courant number. Since the maximum annihilation happens at β = 0, the effect of *Co* to the results accuracy is further as investigated. The coefficient of determination R^2 and average error *E* for comparison between the computed pressure and the actual pressure can be analyzed using Eqs. (16) and (17) respectively. The actual pressure is calculated by taking β = 1 with small *Co* of 0.05.

$$R^2 = 1 - \frac{\sum_{i=1}^{n} \left\{ \left(P_i\right)_{\text{computed}} - \left(P_i\right)_{\text{exact}} \right\}^2}{\sum_{i=1}^{n} \left\{ \left(P_i\right)_{\text{computed}} - P_{\text{average}} \right\}^2} \tag{16}$$

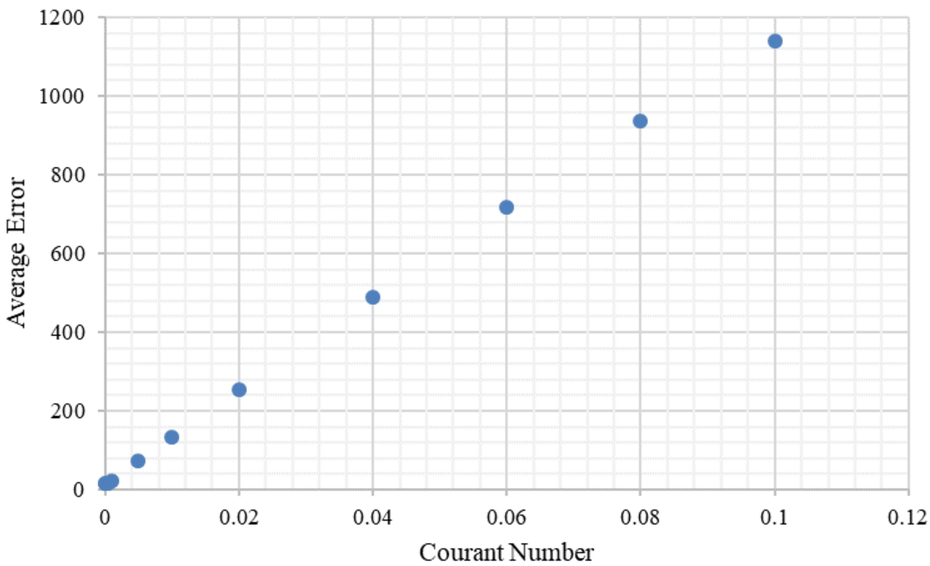
$$\text{Average Error} = \frac{1}{n} \sum_{i=1}^{n} \left| \left(P_i\right)_{\text{computed}} - \left(P_i\right)_{\text{exact}} \right| \tag{17}$$

Figure 4(a) and (b) illustrates the effect of *Co* to the propagated wave pattern and the averaged error. For examination purpose, the spatial domain is zoomed into the distance only up to 6.16 × 10^4 μm, the total time is set only to 150 μs while the grid size is reduced only to 160. The Courant number is reduced until an undamped wave is obtained. *Co* is gradually decreased until the convergence is reached (i.e. the results will not change any more by increment of Courant number). From Figure 4(a) and (b), it is clearly shown that by decreasing the *Co*, the averaged error will be reduced. However, the significant reduction of *Co* is required to obtain a result with R^2 approaches to 1.

In order to ensure the numerical accuracy for every β, *Co* is gradually reduced upon convergence. The maximum *Co* required and the CPU time required to a stable and undamped computation has been computed as in Table 1. The results are obtained with processor of Intel® Core™ i7-8700K CPU @ 3.40 GHz with RAM 32.0 GB. It can be clearly observed that the minimum Courant number required is very high while the CPU time is very expensive when β = 0. Moreover, the algorithm needs to store the field memory up to five time-steps due to the necessity of boundary condition.



(a)



(b)

Fig. 4. Effect of reduction in Courant number to: (a) propagated wave pattern; and (b) average error

Indeed, despite implicit method is unconditionally stable, yet during the solution of hyperbolic equation, it does not give much advantages compared with fully explicit method. In fact, the time step required for implicit solver is much higher than the explicit solver. Moreover, the implementation of the boundary condition is difficult when *Co* has been decreased to deal with the implicit damping. This is possibly the main reason why in most of the available literature working on the improvement of time integration method for linear wave equation, modification based on explicit methods [9-10,19-22] are more preferable. Some modifications on implicit method [11,18,23] have been reported too, yet with the cost of computational complexity.

Table 1
Maximum Courant number for stable computation and CPU time required for different β

β	*Co*	R^2	CPU Time (seconds)
0.0	0.00005	1.00	10701.837
0.1	0.00080	1.00	583.662
0.2	0.00100	1.00	473.085
0.3	0.00200	1.00	240.449
0.4	0.00400	1.00	117.598
0.5	0.00500	1.00	95.535
0.6	0.00600	1.00	79.622
0.7	0.00800	1.00	59.667
0.8	0.01000	1.00	48.148
0.9	0.02000	1.00	23.800
1.0	0.05000	1.00	9.358

The value of *Co* in Table 1 would ensure an undamped wave propagation despite a course grid. However, with the increment of grid resolution, the *Co* can be increased, especially if the value of β is 1.0. Therefore, in solving GTDTM, *Co* independence study is required too besides than mesh independence study, so that the value of *Co* in which the result will not change anymore regardless of smaller value of *Co* can be identified.

5. Concluding Remarks

Generalised Trapezoidal Differencing Time-Marching (GTDTM) Scheme has been developed to solve the hyperbolic wave equation. With the decreasing value of explicit factor β, the resultant waves are oscillating with a damped manner, and upon some distance, the wave will be fully annihilated. To rectify the dissipative issue, the Courant number is reduced, and the maximum Courant number required for each β has been computed for the ease of implementation for GTDTM. Indeed, for efficient computation of hyperbolic wave equation, explicit method is still more favorable compared with implicit method.

Acknowledgement
The authors would like to thank the support from Takasago Research Fund (Vote No: 4B314) from Malaysia-Japan International Institute of Technology, Universiti Teknologi Malaysia and the Pioneer Scientist Incentive Fund (PSIF) with vote number Proj-In-FETBE-038.

References
[1] Khan, N., A. Kalair, N. Abas, and A. Haider. "Review of ocean tidal, wave and thermal energy technologies." *Renewable and Sustainable Energy Reviews* 72 (2017): 590-604.
[2] Tey, Wah Yen, Kiat Moon Lee, Nor Azwadi Che Sidik, and Yutaka Asako. "Delfim-Soares explicit time marching method for modelling of ultrasonic wave in microalgae pre-treatment." In *IOP Conference Series: Earth and Environmental Science*, vol. 268, no. 1, p. 012106. IOP Publishing, 2019.
[3] Umul, Yusuf Ziya. "Relativistic wave equation for a quantum particle with potential energy." *Optik* 180 (2019): 220-225.
[4] Keller, Joseph B., and Michael Miksis. "Bubble oscillations of large amplitude." *The Journal of the Acoustical Society of America* 68, no. 2 (1980): 628-633.
[5] Tey, Wah Yen, Jared Tang Tze Hou, and Hooi Siang Kang. "Numerical Modelling of Convective Wave using Fractional-Step Method." *International Journal of Engineering & Technology* 7, no. 4.25 (2018): 99-103.
[6] Yasui, Kyuichi. (2018). *Bubble Dynamics - Acoustic Cavitation and Bubble Dynamics: Theoretical and Computational Chemistry*, page 37–97.

[7] Lax, Peter D. "Weak solutions of nonlinear hyperbolic equations and their numerical computation." *Communications on pure and applied mathematics* 7, no. 1 (1954): 159-193.
[8] Tannehill, J. C., Anderson, D. A. and Pletcher, R. H. (1997) *Computational Fluid Mechanics and Heat Transfer, 2nd ed*. United States of America: Taylor & Francis, page 112.
[9] Soares Jr, Delfim. "A novel family of explicit time marching techniques for structural dynamics and wave propagation models." *Computer Methods in Applied Mechanics and Engineering* 311 (2016): 838-855.
[10] Grote, Marcus J., and Teodora Mitkova. "High-order explicit local time-stepping methods for damped wave equations." *Journal of Computational and Applied Mathematics* 239 (2013): 270-289.
[11] Soares, Delfim. "An implicit family of time marching procedures with adaptive dissipation control." *Applied Mathematical Modelling* 40, no. 4 (2016): 3325-3341.
[12] Lax, Peter, and Burton Wendroff. "Systems of conservation laws." *Communications on Pure and Applied mathematics* 13, no. 2 (1960): 217-237.
[13] Malkoti, Ajay, Nimisha Vedanti, and Ram Krishna Tiwari. "A highly efficient implicit finite difference scheme for acoustic wave propagation." *Journal of Applied Geophysics* 161 (2019): 204-215.
[14] Gao, Yali, and Liquan Mei. "Implicit–explicit multistep methods for general two-dimensional nonlinear Schrödinger equations." *Applied Numerical Mathematics* 109 (2016): 41-60.
[15] Rieth, A., R. Kovács, and T. Fülöp. "Implicit numerical schemes for generalized heat conduction equations." *International Journal of Heat and Mass Transfer* 126 (2018): 1177-1182.
[16] Müller, G. and Möser, M. (2013) *Handbook of Engineering Acoustics*, Springer Link.
[17] Versteeg, H.K. and Malalasekera, W. (2007). *An Introduction to Computational Fluid Dynamics: The Finite Volume Method*. Pearson Prentice Hall.
[18] Bathe, Klaus-Jürgen, and Mirza M. Irfan Baig. "On a composite implicit time integration procedure for nonlinear dynamics." *Computers & Structures* 83, no. 31-32 (2005): 2513-2524.
[19] Hahn, G. D. "A modified Euler method for dynamic analyses." *International Journal for Numerical Methods in Engineering* 32, no. 5 (1991): 943-955.
[20] Chung, Jintai, and Jang Moo Lee. "A new family of explicit time integration methods for linear and non-linear structural dynamics." *International Journal for Numerical Methods in Engineering* 37, no. 23 (1994): 3961-3976.
[21] Souza, L. A., J. A. M. Carrer, and C. J. Martins. "A fourth order finite difference method applied to elastodynamics: finite element and boundary element formulations." *Structural Engineering and Mechanics* 17, no. 6 (2004): 735-749.
[22] Soares, Delfim, and Francisco Célio de Araújo. "An explicit direct FEM–BEM coupling procedure for nonlinear dynamics." *Engineering Analysis with Boundary Elements* 103 (2019): 94-100.
[23] Chung, J. and Hulbert, G. M. "A time integration algorithm for structural dynamics with improved numerical dissipation: The generalized-α method*." Journal of Applied Mechanics* 60, no. 2 (1993): 371-375.