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Solution of the Porous Medium Equation with Source Terms *via* 4-Point Newton-Explicit Group MKSOR Based on Wave Variable Transformation

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ABSTRACT

This paper presented the 4-point Newton-Explicit Group Modified Kaudd-SOR (4N-EGMKSOR) iterative method combined with the wave variable transformation to solve the porous medium equation with source terms (PMES). The PMES is a nonlinear heat equation having a variety of naturally occurring physical applications, primarily describing processes involving fluid flow, heat transfer and diffusion. However, finding its exact solution can be difficult. Hence, a numerical method is a good option to obtain the approximate solution for the proposed problem. Firstly, we applied the wave variable transformation to the PMES to reduce the PMES into an ordinary differential equation (ODE). This reduction will significantly minimize the computational complexity of the approximate solution for the PMES. Furthermore, the finite difference scheme was used to discretize the reduced form of the PMES, which leads to a nonlinear finite difference approximation equation. The approximate nonlinear equation was then solved using the Newton method, yielding a system of linear equations. In order to solve the formed system of linear equations, the 4N-EGMKSOR was developed and its formulation was derived. In addition, the effectiveness of the proposed method was examined by performing some numerical calculations and the results were compared to the existing iterative methods, i.e., the Newton-Gauss Seidel (N-GS) and the 4-Point Newton- Explicit Group Modified Kaudd SOR (4N-EGMKSOR). Based on the comparison, the 4N-EGMKSOR iterative method proposed in this work is more efficient in getting the converged solution of the PMES compared to N-GS and 4N-EGKSOR iterative methods.

1. Introduction

The porous medium equation with source terms with source terms (PMES) is defined as Eq. (1) [1]:

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$$\frac{\partial u}{\partial t} = a\frac{\partial}{\partial x}\left(u^m \frac{\partial u}{\partial x}\right) + bu^r \tag{1}$$

Such, that a and b are constants, whereas m and r are known rational numbers. This equation is useful in describing various physical phenomena in real-world problems. For instance, it can model population pressure in biological systems and the unsteady heat transfer in the quiescent medium [2]. Also, the PMES can represent a more realistic population dynamics modelling and wound healing process [3].

Finding the exact solution to the PMES can be complicated. Hence, an approximate solution is needed to understand the proposed problem. To obtain an approximate solution, the numerical and analytical methods are the feasible options. Previously, some researchers proposed a few methods to solve the PMES using an analytical or numerical approach. For instance, analytical methods such as the Adomian decomposition method (ADM) employed by Pamuk [4], the total variation diminishing third-order Runge-Kutta scheme by Sari [5], the homotopy perturbation method (HPM) by Biazar *et al.,* [1] and the combination of HPM with He's polynomial by Saberi *et al.,* [6] have been utilized to solve the PMES. However, a thorough search of the relevant literature related to the numerical approach for solving the PMES yielded only one related article, namely a quarter-sweep finite difference scheme paired with the modified successive over-relaxation iterative method [7].

Motivated by Chew *et al.*, [7] prior research into the approximate solution to the PMES and the lack of a numerical approach to solving the PMES, we introduced in this work a numerical method for solving the PMES. The method is called the 4-Point Newton-Explicit Group Modified Kaudd Successive Over Relaxation iterative method (4N-EGMKSOR) combined with the wave variable transformation. This method basically reduces the PMES into an ordinary differential equation (ODE) first, then continues solving the resultant ODE numerically. Consequently, the computation costs of the approximate solution will be significantly minimized, which means an improvement in the efficiency of the proposed method.

The theory of traveling waves has been studied by some authors to examine the existence of traveling waves to PMES [8,9]. Additionally, several travelling wave applications are also discussed in the literature by some authors in literatures [10-15]. For example, Ghazaryan *et al.,* [10] applied the travelling wave variable to the Korteweg-deVries (KdV) equation to find its travelling wave solution. The authors found that the travelling wave solution to the KdV equation has a form of traveling plus solution for any positive value of wave speed, $c > 0$ and the known initial value of the wave equation for KdV, which is $\xi_0 = 2c/c$. On the other hand, Bibi *et al.,* [11] also applied the travelling wave variable to the Korteweg-deVries (KdV) equation to reduce it into a nonlinear ODE. Then, the reduced KdV equation was solved using the Sine-Cosine method and some computerized symbolic calculations. Moreover, Mansour [13] performed a travelling wave analysis of the nonlinear doubly degenerate reaction-diffusion equation to find out the travelling wave behaviour of its solutions. The author found out that the behaviour of the travelling wave solution for the equation was sharp type and smooth. The previous years, Mansour [15] also conducted a travelling wave analysis to the density-dependent diffusion Nagumo equation to show the existence of travelling wave solution to that equation. This analysis was conducted using a nonlinear dynamical system approach and the author concluded that the equation admits the travelling wave solution with a sharp front type. Consequently, the density-dependent diffusion Nagumo equation can be simplified to its corresponding travelling wave equation and the solution to the reduced equation can be determined through numerical computation.

The method that was proposed draws significant inspiration from the method described by Ali *et al.,* [16,17], which effectively utilised wave variable transformation to solve linear partial differential

equations. Nevertheless, we expanded upon their research by applying it to a nonlinear partial differential equation (PDE) that exhibits nonlinearity.

Section 2 covers the process of developing the finite difference approximation of the PMES, while section 3 focuses on the implementation of the 4N-EGMKSOR iterative approach. Subsequently, we showcased the quantitative outcomes in section 4 and provided our assessments of our research in section 5.

2. Development of the Approximation Equation

In this section, the approximate equation of the PMES will be derived by using the finite difference scheme according to its traveling wave equation. To begin with, the PMES in Eq. (1) can be rewritten as Eq. (2):

$$\frac{\partial u}{\partial t} = au^m \frac{\partial^2 u}{\partial x^2} + amu^{m-1}\left(\frac{\partial u}{\partial x}\right)^2 + bu^r \ , \tag{2}$$

Where, $0 \le x \le 1$ and $t > 0$. Next, Eq. (2) is converted into its appropriate travelling wave equation using the travelling wave transformation. This transformation is a general transformation under the similarity transformation [14]. The travelling wave transformation introduced a variable given by $\xi = x - ct$ where c is a constant representing a wave velocity [10,14]. Furthermore, $u(x,t) = u(\xi)$ will become the solution to the corresponding travelling wave equation. Now, some chain rules are applied on ξ to generate the appropriate substitutions for the derivatives in Eq. (2) which yield Eq. (3) [18]:

$$\begin{cases} u_t = -cu' \\ u_x = u' \\ u_{xx} = u'' \end{cases} . \tag{3}$$

Substituting Eq. (3) into Eq. (2), the PMES then transformed into a corresponding nonlinear ODE, i.e Eq. (4):

$$-c\frac{du}{d\xi} = au^m \frac{d^2 u}{d\xi^2} + amu^{m-1}\left(\frac{du}{d\xi}\right)^2 + bu^r \ , \tag{4}$$

With, $\xi_I \le \xi \le \xi_F$, where $\xi_I = -ct$ and $\xi_F = 1 - ct$ and $t > 0$. Observe that Eq. (4) becomes a nonlinear second-order ODE that depends only on one independent variable ξ. Therefore, we are now dealing with an ODE instead of a PDE. Hereafter, Eq. (4) is discretized using the second-order central difference scheme, which is in Eqs. (5) and (6):

$$u'(\xi_i) \approx \frac{u_{i+1} - u_{i-1}}{2h} \ , \tag{5}$$

and

$$u''(\xi_i) \approx \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} \quad . \tag{6}$$

The subscript i is assigned the values $i = 1, 2, 3, \cdots, (M-1)$, whereas $h = \big((1-ct) - (-ct)\big)/M$ and M being the number of subintervals in ξ direction. By substituting Eq. (5) and Eq. (6) into Eq. (4), Eq. (4) is transformed into Eq. (7):

$$\alpha\left(u_{i+1} - u_{i-1}\right) = \beta u_i^{\,m}\left(u_{i+1} - 2u_i + u_{i-1}\right) + \gamma m u_i^{\,m-1}\left(u_{i+1} - u_{i-1}\right)^2 + b u_i^{\,r} \quad , \tag{7}$$

Such, that $\alpha = -c/2h$, $\beta = a/h^2$ and $\gamma = a/4h^2$. The Eq. (7) represents the finite difference approximation equation to Eq. (2). Furthermore, Eq. (7) also may be expressed as a nonlinear function as in Eq. (8):

$$f_i(\underset{\sim}{u}) = \alpha\left(u_{i+1} - u_{i-1}\right) - \beta u_i^{\,m}\left(u_{i+1} - 2u_i + u_{i-1}\right) - \gamma m u_i^{\,m-1}\left(u_{i+1} - u_{i-1}\right)^2 - b u_i^{\,r} \quad , \tag{8}$$

for $i = 1, 2, 3, \cdots, (M-1)$. Following this, Eq. (8) was subjected to the Newton method [19] in order to derive its corresponding linear system, which is given by Eq. (9):

$$J\left(\underset{\sim}{u}^{(k)}\right)\Delta\underset{\sim}{u}^{(k)} = -\underset{\sim}{f}\left(\underset{\sim}{u}^{(k)}\right) , \tag{9}$$

Where, $\underset{\sim}{u} = (u_1, u_2, \cdots, u_{M-1})^T$, $\Delta\underset{\sim}{u} = (\Delta u_1, \Delta u_2, \cdots, \Delta u_{M-1})^T$, $\underset{\sim}{f} = (f_1, f_2, \cdots, f_{M-1})^T$ and k is the index of iteration. The Jacobian matrix in Eq. (9) is defined as in Eq. (10):

$$J(\underset{\sim}{u}^{(k)}) = \begin{pmatrix} f_{1u_1} & f_{1u_2} & \cdots & f_{1u_{M-1}} \\ f_{2u_1} & f_{2u_2} & \cdots & f_{2u_{M-1}} \\ \vdots & \vdots & \ddots & \vdots \\ f_{M-1u_1} & f_{M-1u_2} & \cdots & f_{M-1u_{M-1}} \end{pmatrix}^{(k)}_{(M-1)\times(M-1)} \quad , \tag{10}$$

Where, we defined $f_{iu_j} = \partial f_i / \partial u_j$. Finally, we compute the approximate solutions $\underset{\sim}{u}^{(k)}$ *via* the Eq. (11):

$$\underset{\sim}{u}^{(k+1)} = \Delta\underset{\sim}{u}^{(k)} + \underset{\sim}{u}^{(k)} \quad . \tag{11}$$

3. Implementation of the 4N-EGMKSOR Iteration

To solve the system of linear equations in Eq. (9), we employed the 4-Point Explicit Group MKSOR iterative method, which is the combination of the Explicit Group [20] and MKSOR methods [21,22]. The MKSOR method implements the red-black ordering strategy [23] with two relaxation parameters [24], ω_1 and ω_2 , in the domain of $\mathbb{R} \setminus [-2, 0]$. Thus, the formulation of the MKSOR iterative method based on Eq. (9) is expressed as in Eq. (12) [24,25]:

$$\Delta u_i^{(k+1)} = \frac{1}{1+\omega_1}\left[\Delta u_i^{(k)} + \frac{\omega_1}{f_{iu_i}}\left((-f_i) - \sum_{j=1}^{i-1} f_{iu_j}\Delta u_j^{(k+1)} - \sum_{j=i+1}^{M-1} f_{iu_j}\Delta u_j^{(k)}\right)\right], \tag{12}$$

for $i = 2,4,6,\cdots,(M-2)$.

The Jacobian matrix in Eq. (9) is large scale depending on the value of M. It is also a sparse matrix, as most of the entries are zero. In fact, it can be expressed as a tridiagonal matrix whose value is subject to the condition that:

$$J(\underset{\sim}{u}^{(k)}) = \begin{pmatrix} b_1 & c_1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ a_2 & b_2 & c_2 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & a_3 & b_3 & c_3 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & a_4 & b_4 & c_4 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & a_{M-2} & b_{M-2} & c_{M-2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & a_{M-1} & b_{M-1} \end{pmatrix}^{(k)}_{(M-1)\times(M-1)}, \tag{13}$$

Where, $a_i = \partial f_i / \partial u_{i-1}$, $b_i = \partial f_i / \partial u_i$ and $c_i = \partial f_i / \partial u_{i+1}$ for $i = 1,2,3,\cdots,(M-1)$. Therefore, we intended to facilitate the iteration process of the MKSOR iterative method by using the 4–point Explicit Group iterative method that was proposed by Evans to solve a large, sparse linear system [20].

The 4-point Explicit Group iterative method is constructed based on the linear system Eq. (9), resulting in a solution domain comprising several completed groups of four-points and ungroup points. The ungrouped points are dealt with utilizing a three-point iteration scheme [20]. Figure 1 illustrates the groups of four-points enclosed in the red-coloured rectangle, the ungrouped points enclosed in the green-coloured rectangle and blue dots are the boundary conditions for the PMES expressed in Eq. (4). Furthermore, in addition, Eq. (7) is used to link the grid points in Figure 1 with h increment.

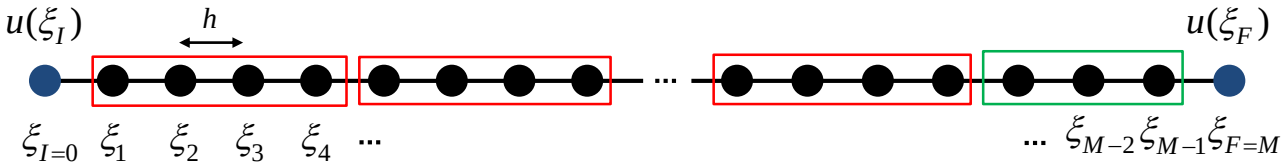


Fig. 1. Finite grid network of the 4-point explicit group iterative method for traveling wave equation for the PMES

Now, the 4-point EG iterative method for any completed group is given by the following Eq. (14) [26];

$$\begin{pmatrix} \Delta u_p \\ \Delta u_{p+1} \\ \Delta u_{p+2} \\ \Delta u_{p+3} \end{pmatrix}^{(k+1)} = \begin{pmatrix} b_p & c_p & 0 & 0 \\ a_{p+1} & b_{p+1} & c_{p+1} & 0 \\ 0 & a_{p+2} & b_{p+2} & c_{p+2} \\ 0 & 0 & a_{p+3} & b_{p+3} \end{pmatrix}^{-1} \begin{pmatrix} s_p \\ s_{p+1} \\ s_{p+2} \\ s_{p+3} \end{pmatrix}, \tag{14}$$

Where, $p=1,5,\cdots,(M-7)$, $s_p=-f_p-a_p\Delta u_{p-1}^{(k)}$, $s_{p+1}=-f_{p+1}$, $s_{p+2}=-f_{p+2}$ and $s_{p+3}=-f_{p+3}-a_{p+3}\Delta u_{p+4}^{(k)}$. To solve the ungroup points, we consider the following iterative method in Eq. (15):

$$\begin{pmatrix}\Delta u_p\\ \Delta u_{p+1}\\ \Delta u_{p+2}\end{pmatrix}^{(k+1)}=\begin{pmatrix}b_p & c_p & 0\\ a_{p+1} & b_{p+1} & c_{p+1}\\ 0 & a_{p+2} & b_{p+2}\end{pmatrix}^{-1}\begin{pmatrix}s_p\\ s_{p+1}\\ s_{p+2}\end{pmatrix}, \tag{15}$$

Where, $p=(M-3)$, $s_p=-f_p-a_p\Delta u_{p-1}^{(k)}$, $s_{p+1}=-f_{p+1}$ and $s_{p+2}=-f_{p+2}-c_{p+2}\Delta u_{p+3}^{(k)}$. Thus, by referring to Figure 1 as a guide and Eq. (12) until Eq. (14), the 4-point Explicit Group KSOR iterative method may be derived as in Eq. (16):

$$\begin{pmatrix}\Delta u_p\\ \Delta u_{p+1}\\ \Delta u_{p+2}\\ \Delta u_{p+3}\end{pmatrix}^{(k+1)}=\frac{1}{1+\omega_1}\begin{pmatrix}\Delta u_p\\ \Delta u_{p+1}\\ \Delta u_{p+2}\\ \Delta u_{p+3}\end{pmatrix}^{(k)}+\frac{\omega_1}{1+\omega_1}\begin{pmatrix}b_p & c_p & 0 & 0\\ a_{p+1} & b_{p+1} & c_{p+1} & 0\\ 0 & a_{p+2} & b_{p+2} & c_{p+2}\\ 0 & 0 & a_{p+3} & b_{p+3}\end{pmatrix}^{-1}\begin{pmatrix}s_p\\ s_{p+1}\\ s_{p+2}\\ s_{p+3}\end{pmatrix}, \tag{16}$$

for $p=1,9,17\cdots,(M-7)$ as in Eq. (17):

$$\begin{pmatrix}\Delta u_p\\ \Delta u_{p+1}\\ \Delta u_{p+2}\\ \Delta u_{p+3}\end{pmatrix}^{(k+1)}=\frac{1}{1+\omega_2}\begin{pmatrix}\Delta u_p\\ \Delta u_{p+1}\\ \Delta u_{p+2}\\ \Delta u_{p+3}\end{pmatrix}^{(k)}+\frac{\omega_2}{1+\omega_2}\begin{pmatrix}b_p & c_p & 0 & 0\\ a_{p+1} & b_{p+1} & c_{p+1} & 0\\ 0 & a_{p+2} & b_{p+2} & c_{p+2}\\ 0 & 0 & a_{p+3} & b_{p+3}\end{pmatrix}^{-1}\begin{pmatrix}s_p\\ s_{p+1}\\ s_{p+2}\\ s_{p+3}\end{pmatrix}, \tag{17}$$

for $p=5,13,21,\cdots,(M-11)$. Last but not least, the ungrouped points are computed using Eq. (18):

$$\begin{pmatrix}\Delta u_p\\ \Delta u_{p+1}\\ \Delta u_{p+2}\end{pmatrix}^{(k+1)}=\frac{1}{1+\omega_2}\begin{pmatrix}\Delta u_p\\ \Delta u_{p+1}\\ \Delta u_{p+2}\end{pmatrix}^{(k)}+\frac{\omega_2}{1+\omega_2}\begin{pmatrix}b_p & c_p & 0\\ a_{p+1} & b_{p+1} & c_{p+1}\\ 0 & a_{p+2} & b_{p+2}\end{pmatrix}^{-1}\begin{pmatrix}s_p\\ s_{p+1}\\ s_{p+2}\end{pmatrix} \tag{18}$$

for $p=(M-3)$. In addition, we presented the general algorithm for the 4-point Newton-Explicit Group MKSOR iterative method to solve the PMES, as shown in Algorithm 1.

Algorithm 1: 4-Point Newton-Explicit Group MKSOR iterative method:

i. Specify the boundary conditions i.e $u(\xi_I)$ and $u(\xi_F)$.
ii. Fix the value of $u^{(0)}=\left[u(\xi_I)+u(\xi_F)\right]/2$, $\Delta\underset{\sim}{u}^{(0)}=0.0$, $\varepsilon_u=10^{-10}$ and $\varepsilon_{\Delta u}=10^{-10}$.
iii. Set $\Delta u^{(0)}=0$.
iv. Calculate J and $-\underset{\sim}{f}$.
v. Iterate Eq. (17), Eq. (18) and Eq. (19).
vi. Verify whether $\left\|\Delta\underset{\sim}{u}^{(k+1)}-\Delta\underset{\sim}{u}^{(k)}\right\|\le\varepsilon_{\Delta u}$. If the statement is true, then continue to step vii.

Alternatively, perform step v again.

vii. Compute $\underset{\sim}{u}^{(k+1)} = \Delta\underset{\sim}{u}^{(k)} + \underset{\sim}{u}^{(k)}$.

viii. Determine whether $\left\|\underset{\sim}{u}^{(k+1)} - \underset{\sim}{u}^{(k)}\right\| \le \varepsilon_u$. If the condition is true, then show the approximate solutions for PMES, i.e, $\underset{\sim}{u}^{(k+1)} = \underset{\sim}{u}$. Otherwise, repeat step iii until vii.

To find the optimum values for the relaxation parameters ω_1 and ω_2, a computer program is run with different values of ω until the least number of iterations is attained.

4. Numerical Results

In this section, we examine the effectiveness of the 4-Point Newton-EGMKSOR (4N-EGMKSOR) method by considering the number of iterations (k), the execution time (measured in seconds) (t) and the highest absolute error (E) produced when solving several problems [27]. Furthermore, the 4-Point Newton-Explicit Group (4N-EG) and 4-Point Newton-Explicit Group KSOR (4N-EGKSOR) iterative procedures are recognized as standard references. Regarding the tolerance error, $\varepsilon = 10^{-10}$ is used to assess the convergence of the solution for different sizes of linear systems [28], specifically for sizes $M = 256, 512, 1024, 2048$ and 4096. The problems of interest are as follows:

4.1 Problem 1 [9]

Set $m = -1$ and $r = 2$ in Eq. (1). Thus, the equation becomes as such in Eq. (19):

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(u^{-1}\frac{\partial u}{\partial x}\right) + u^2 \ . \tag{19}$$

This equation is a quasilinear fast diffusion equation with a quadratic reaction component. To verify our calculation, the exact solution given by Polyanin *et al.,* [9] was utilized as a comparison, namely shown in Eq. (20):

$$u(x,t) = \left(\frac{(x+C_1)^2}{2t} + C_2 t - 2t\ln|t|\right)^{-1}, \quad t \neq 0 \ , \tag{20}$$

Where, C_1 and C_2 are arbitrary constants. In this implementation, C_1 and C_2 have been set to 0.35 and 1.35, respectively.

4.2 Problem 2 [9]

Consider an equation that describes the classical case of a gravity current in the air. Eq. (1) becomes Eq. (21):

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(u^{-1}\frac{\partial u}{\partial x}\right) + u^2 \ . \tag{21}$$

Polyanin *et al.,* [9] gave a particular solution for this equation, which is shown in Eq. (22):

$$u(x,t)=\left(Ae^{\frac{2bmt}{m+2}}-\frac{bm^2(x+B)^2}{2a(m+2)}\right)^{\frac{1}{m}}, \tag{22}$$

Where, A and B are arbitrary constants. For this problem, the value for A, a, B and b are set to be 1.35, 1.0, 0.35 and 1.0, respectively.

4.3 Problem 3 [4]

Taking $m=1$ and $r=0$ in Eq. (1) then, it becomes Eq. (23):

$$\frac{\partial u}{\partial t}=a\frac{\partial}{\partial x}\left(u\frac{\partial u}{\partial x}\right)+b, \tag{23}$$

Which, is the heat conduction equation with source terms. The exact solution for this equation is $u(x,t)=x+(a+b)t$, where a and b are set as 1 and -3, respectively.

4.4 Problem 4 [3]

Let us take $m=2$, $r=1$ in Eq. (1), an equation representing a slow particle diffusion on a fresh membrane. Therefore, Eq. (1) becomes Eq. (24):

$$\frac{\partial u}{\partial t}=\frac{\partial}{\partial x}\left(u^2\frac{\partial u}{\partial x}\right)-u\ . \tag{24}$$

The exact solution for this problem provided by Chew *et al.,* [3] was utilized to verify the accuracy of the numerical solutions, which is shown as in Eq. (25):

$$u(x,t)=\left(x-\frac{2}{\beta}\right)e^{\beta t}+\left(\frac{2}{\beta}\right)e^{2\beta t}, \tag{25}$$

Where, β is an arbitrary constant that has been set to -1.

The numerical computations were performed using a laptop computer with an Intel(R) Core i7-6500U CPU running at 2.60GHz and 8 GB of RAM. All numerical results of this experiment are tabulated in Table 1 until Table 5. In addition, the percentage of reduction listed in Table 5 was computed using the expression in Eq. (26) [29]:

$$\Delta k\%=\left|\frac{k_{N-EG}-k_{4N-EGMKSOR}}{k_{N-EG}}\right|\times 100\%, \tag{26}$$

for the number of iterations, Eq. (27) is produced:

$$\Delta t\% = \left| \frac{t_{N-EG} - t_{4N-EGMKSOR}}{t_{N-EG}} \right| \times 100\% \ , \tag{27}$$

for the computational time. On top of that, the maximum absolute error can be determined by the following Eq. (28) [19]:

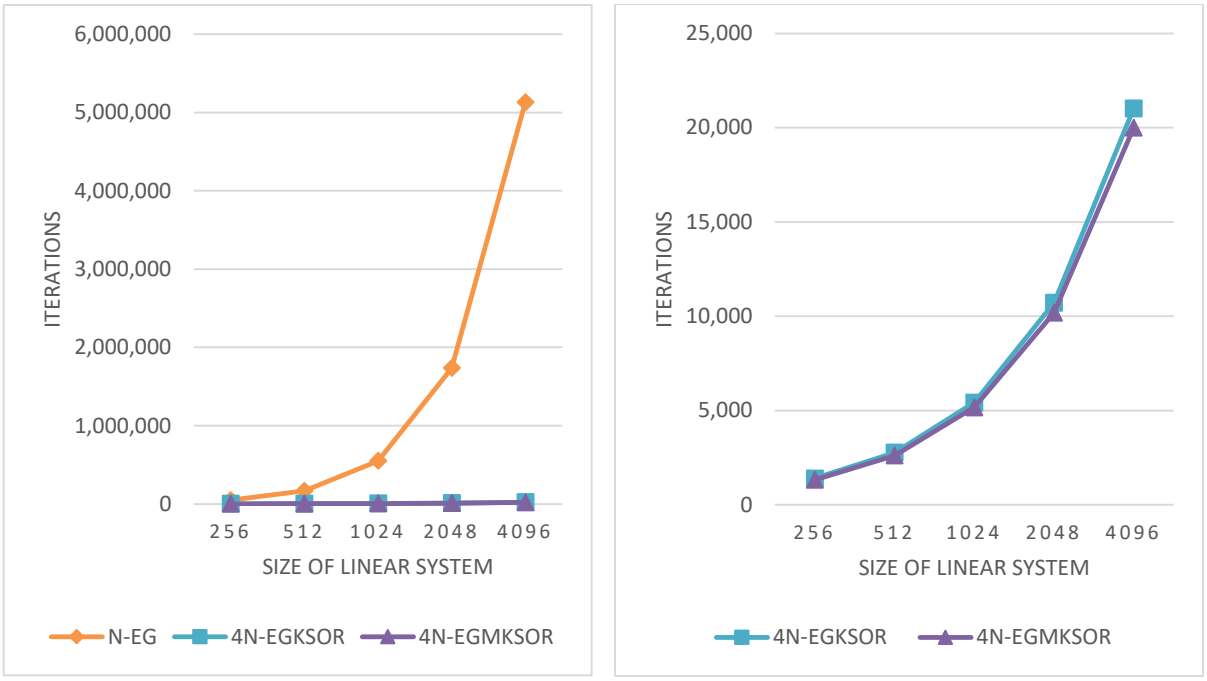
$$E = \max_{i \in [1, M-1]} \left\| \underset{\sim}{u}^{(k+1)} - \underset{\sim}{u}^{(k)} \right\| \ . \tag{28}$$

Based on Table 1 to 4, the 4N-EGMKSOR iterative method required fewer iteration numbers (k) and less computation time (t) at every size linear system (M) compared to the 4N-EG and 4N-EGKSOR iterative methods. This comparison is also presented graphically in Figure 2 to 9 for each problem, where the graph for the 4N-EGMKSOR lies below the graph of the 4N-EG and 4N-EGKSOR. Hence, it indicates that the values on the graph for the 4N-EGMKSOR are consistently lower than the values on the graph for the 4N-EG and 4N-EGKSOR.

In Table 5, the 4N-EGMKSOR iterative method can minimize the iteration numbers against the 4N-EG to at least 97.27%, 98.43%, 99.06%, 99.41% and 99.61% for the linear systems of size 256, 512, 1024, 2048 and 4096, respectively. Furthermore, the 4N-EGMKSOR iterative method reduces the computational time against the 4N-EG to at least 55.88%, 97.47%, 98.79%, 99.24% and 99.51% for the linear systems of size 256, 512, 1024, 2048 and 4096, respectively. The reductions are achievable as a result of the use of two optimal relaxation parameters in the computation, as indicated in Table 1 to 4. These parameters expedite the convergence rate towards an approximation of the PMES solution. In terms of accuracy, all methods are in good agreement. When compared to 4N-EG, the accuracy of the iterative methods 4N-EGKSOR and 4N-EGMKSOR improves marginally. Therefore, based on the experimental results, the 4N-EGMKSOR iterative method is more efficient in solving the PMES compared to the two benchmarks.

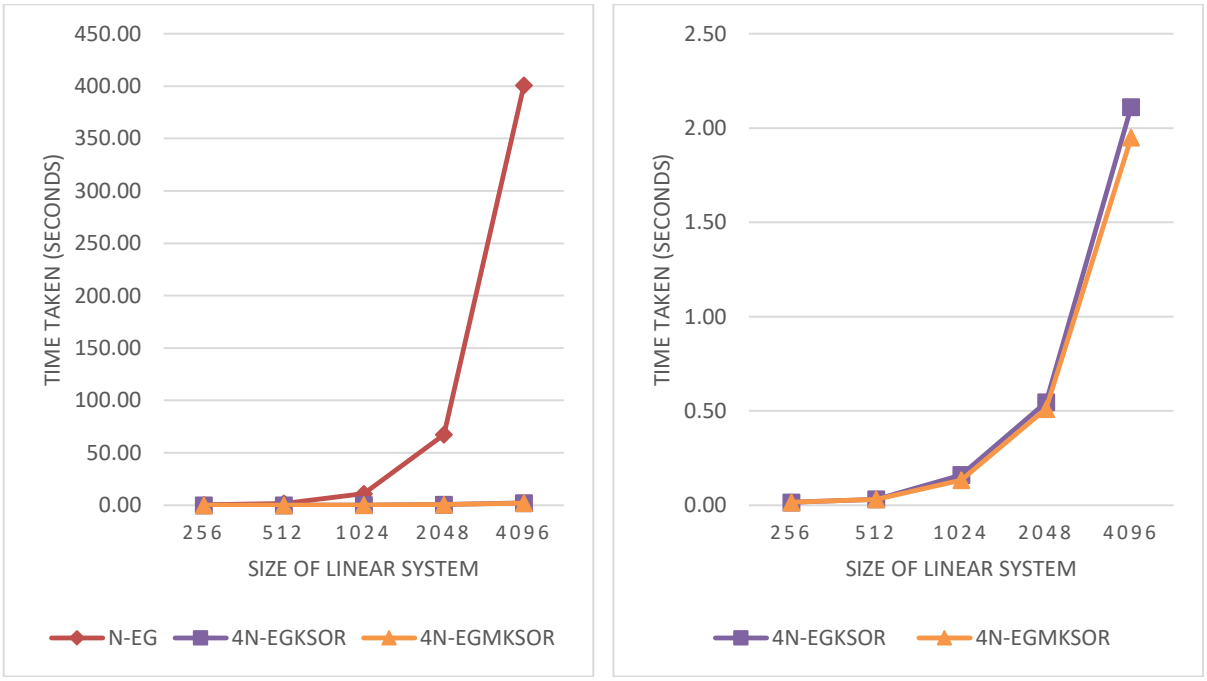
Table 1
The number of iterations, time taken and maximum absolute error generated by N-GS, 4N-EGKSOR and 4N-EGMKSOR at different grid sizes for Problem 1

M	Method	ω_1	ω_2	k	t	E
256	N-EG			49218	0.24	1.8681×10^{-04}
	4N-EGKSOR	-2.0462		1399	0.02	1.8709×10^{-04}
	4N-EGMKSOR	-2.0489	-2.0493	1321	0.02	1.8709×10^{-04}
512	N-EG			166788	1.68	1.8727×10^{-04}
	4N-EGKSOR	-2.0231		2762	0.03	1.8707×10^{-04}
	4N-EGMKSOR	-2.0244	-2.0247	2613	0.03	1.8706×10^{-04}
1024	N-EG			549995	10.95	1.8891×10^{-04}
	4N-EGKSOR	-2.0117		5420	0.16	1.8706×10^{-04}
	4N-EGMKSOR	-2.0123	-2.0125	5164	0.13	1.8706×10^{-04}
2048	N-EG			1734570	67.41	1.9548×10^{-04}
	4N-EGKSOR	-2.0059		10709	0.55	1.8706×10^{-04}
	4N-EGMKSOR	-2.0062	-2.0063	10181	0.51	1.8706×10^{-04}
4096	N-EG			5129203	400.87	2.2224×10^{-04}
	4N-EGKSOR	-2.0030		21024	2.11	1.8706×10^{-04}
	4N-EGMKSOR	-2.0032	-2.0033	20007	1.95	1.8706×10^{-04}



(a) (b)

Fig. 2. (a) Shows the difference in iteration counts against the size of linear system for N-EG, 4N-EGKSOR and 4N-EGMKSOR. (b) Shows the close up look of the difference in iteration counts against the size of linear system for the 4N-EGKSOR and 4N-EGMKSOR

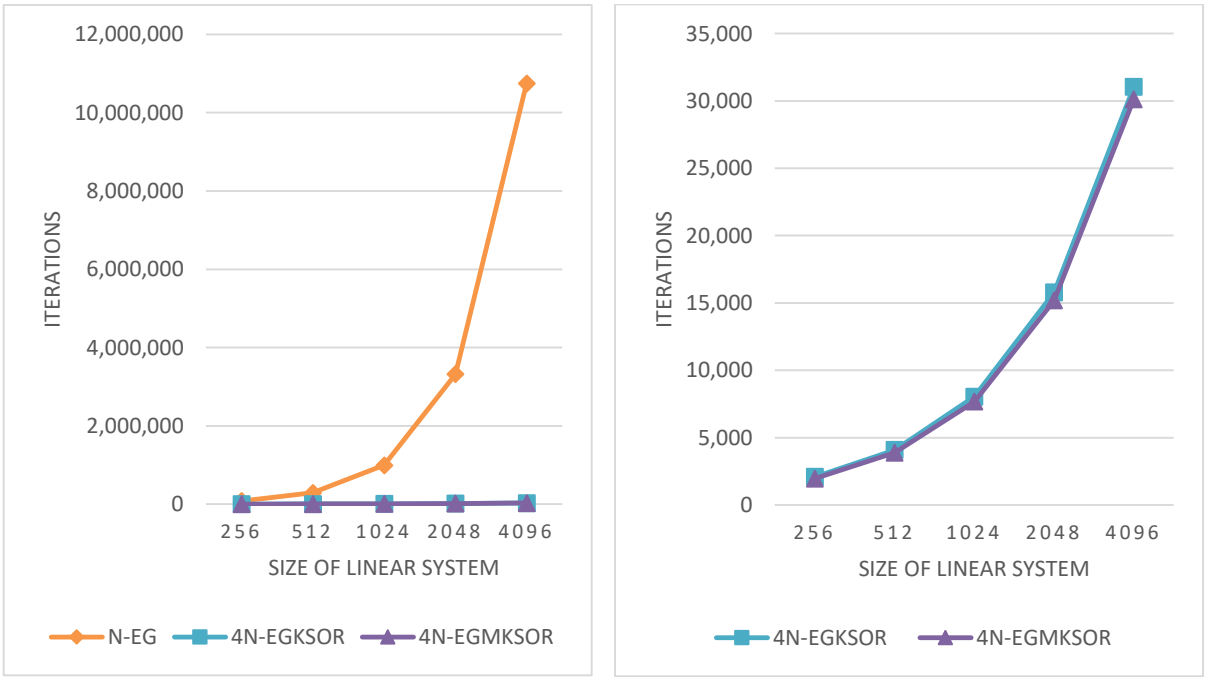


(a) (b)

Fig. 3. (a) Shows the difference in time taken (*seconds*) against the size of linear system for N-EG, 4N-EGKSOR and 4N-EGMKSOR. (b) Shows the close up look of the difference in time taken (*seconds*) against the size of linear system for the 4N-EGKSOR and 4N-EGMKSOR

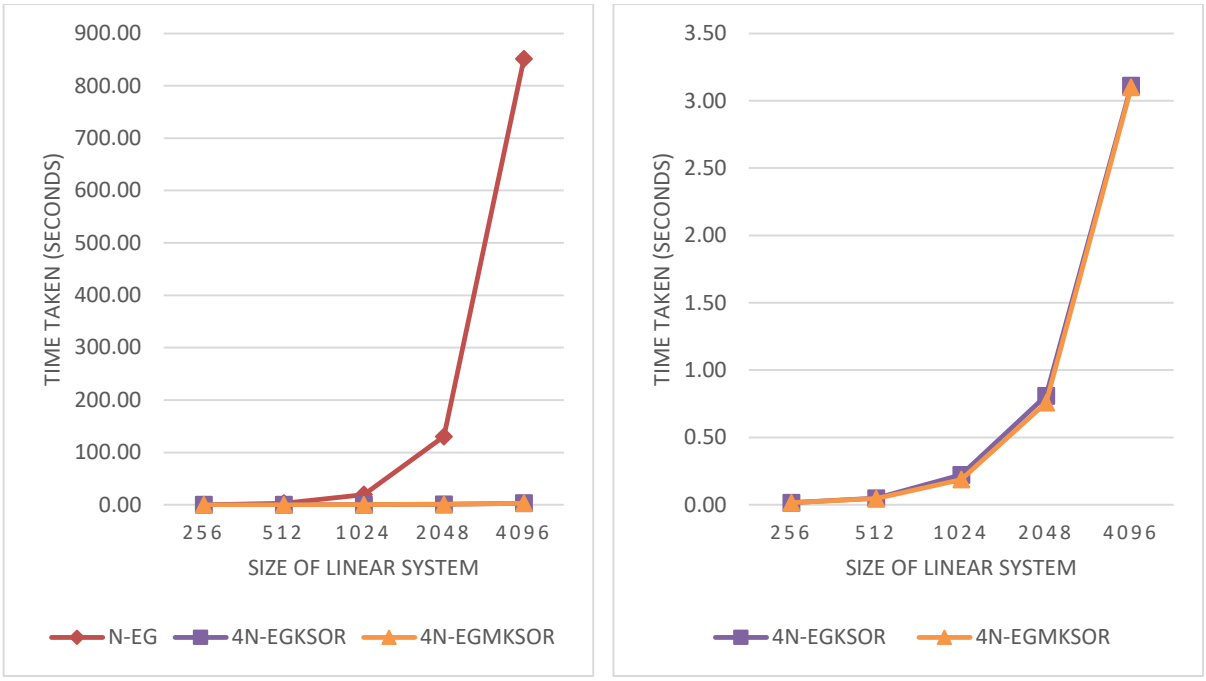
Table 2
The number of iterations, time taken and maximum absolute error generated by N-GS, 4N-EGKSOR and 4N-EGMKSOR at different grid sizes for Problem 2

M	Method	ω_1	ω_2	k	t	E
256	N-EG			82738	0.03	1.6583×10^{-03}
	4N-EGKSOR	-2.0528		2079	0.02	1.6583×10^{-03}
	4N-EGMKSOR	-2.0531	-2.0535	1951	0.02	1.6583×10^{-03}
512	N-EG			289268	3.12	1.6588×10^{-03}
	4N-EGKSOR	-2.0535		4076	0.05	1.6584×10^{-03}
	4N-EGMKSOR	-2.0535	-2.0265	3875	0.05	1.6584×10^{-03}
1024	N-EG			992073	19.62	1.6599×10^{-03}
	4N-EGKSOR	-2.0130		8035	0.22	1.6584×10^{-03}
	4N-EGMKSOR	-2.0133	-2.0133	7673	0.19	1.6584×10^{-03}
2048	N-EG			3318699	130.43	1.6644×10^{-03}
	4N-EGKSOR	-2.0065		15779	0.81	1.6584×10^{-03}
	4N-EGMKSOR	-2.0067	-2.0067	15185	0.76	1.6584×10^{-03}
4096	N-EG			10746838	851.63	1.6880×10^{-03}
	4N-EGKSOR	-2.0033		31028	3.11	1.6584×10^{-03}
	4N-EGMKSOR	-2.0034	-2.0034	30124	3.10	1.6584×10^{-03}



(a) (b)

Fig. 4. (a) Shows the difference in iteration counts against the size of linear system for N-EG, 4N-EGKSOR and 4N-EGMKSOR. (b) Shows the close up look of the difference in iteration counts against the size of linear system for the 4N-EGKSOR and 4N-EGMKSOR

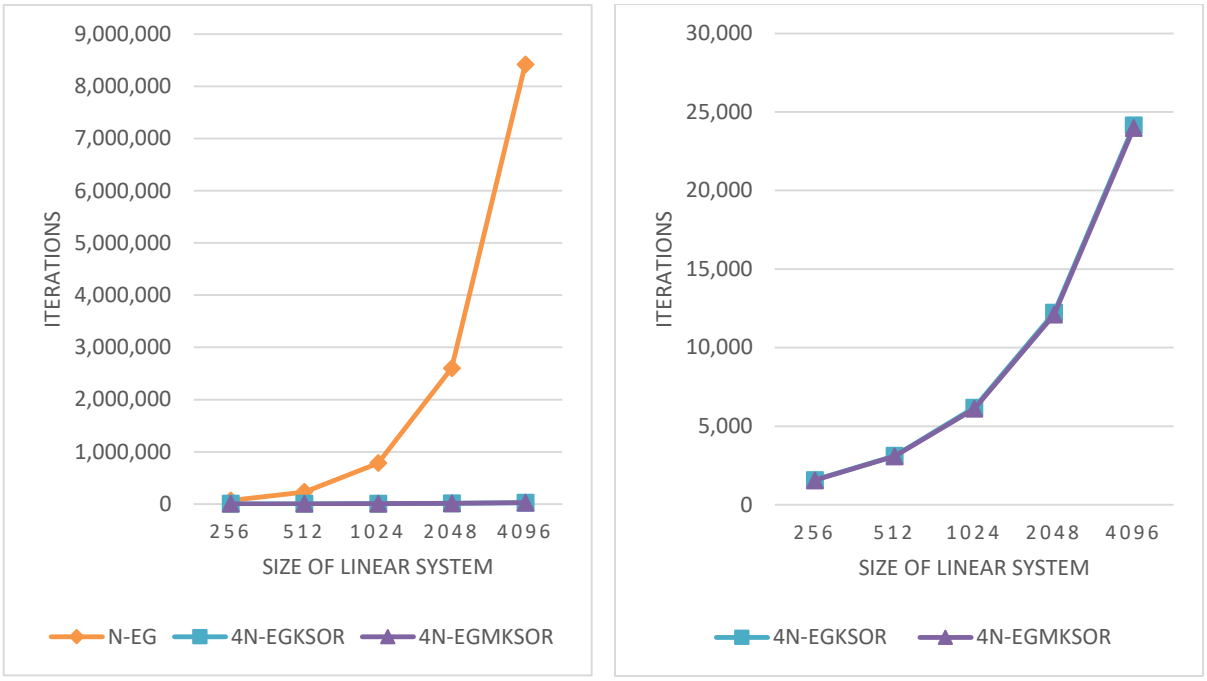


(a) (b)

Fig. 5. (a) Shows the difference in time taken (*seconds*) against the size of linear system for N-EG, 4N-EGKSOR and 4N-EGMKSOR. (b) Shows the close up look of the difference in time taken (*seconds*) against the size of linear system for the 4N-EGKSOR and 4N-EGMKSOR

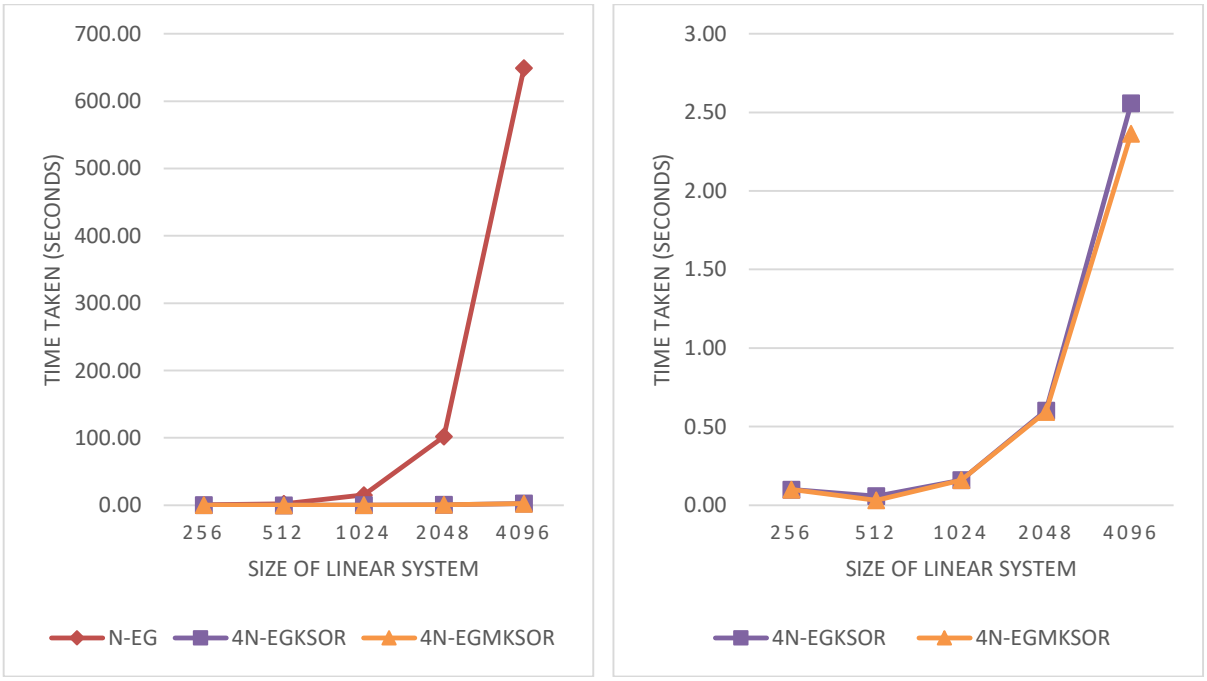
Table 3
The number of iterations, time taken and maximum absolute error generated by N-GS, 4N-EGKSOR and 4N-EGMKSOR at different grid sizes for Problem 3

M	Method	ω_1	ω_2	k	t	E
256	N-EG			65556	0.31	1.5132×10^{-07}
	4N-EGKSOR	-2.0518		1574	0.00	9.5551×10^{-11}
	4N-EGMKSOR	-2.0522	-2.0520	1559	0.00	6.6120×10^{-11}
512	N-EG			228564	2.20	6.0648×10^{-07}
	4N-EGKSOR	-2.0258		3118	0.06	1.1923×10^{-10}
	4N-EGMKSOR	-2.0260	-2.0260	3089	0.03	1.0040×10^{-10}
1024	N-EG			780456	15.05	2.4307×10^{-07}
	4N-EGKSOR	-2.0129		6186	0.16	1.3166×10^{-10}
	4N-EGMKSOR	-2.0131	-2.0129	6108	0.16	1.2480×10^{-10}
2048	N-EG			2600860	101.97	9.9434×10^{-07}
	4N-EGKSOR	-2.0065		12222	0.60	1.9517×10^{-10}
	4N-EGMKSOR	-2.0066	-2.0065	12099	0.60	1.8303×10^{-10}
4096	N-EG			8418880	649.08	4.4469×10^{-07}
	4N-EGKSOR	-2.0033		24144	2.56	3.2026×10^{-10}
	4N-EGMKSOR	-2.0033	-2.0033	23978	2.37	3.1766×10^{-10}



(a) (b)

Fig. 6. (a) Shows the difference in iteration counts against the size of linear system for N-EG, 4N-EGKSOR and 4N-EGMKSOR. (b) Shows the close up look of the difference in iteration counts against the size of linear system for the 4N-EGKSOR and 4N-EGMKSOR

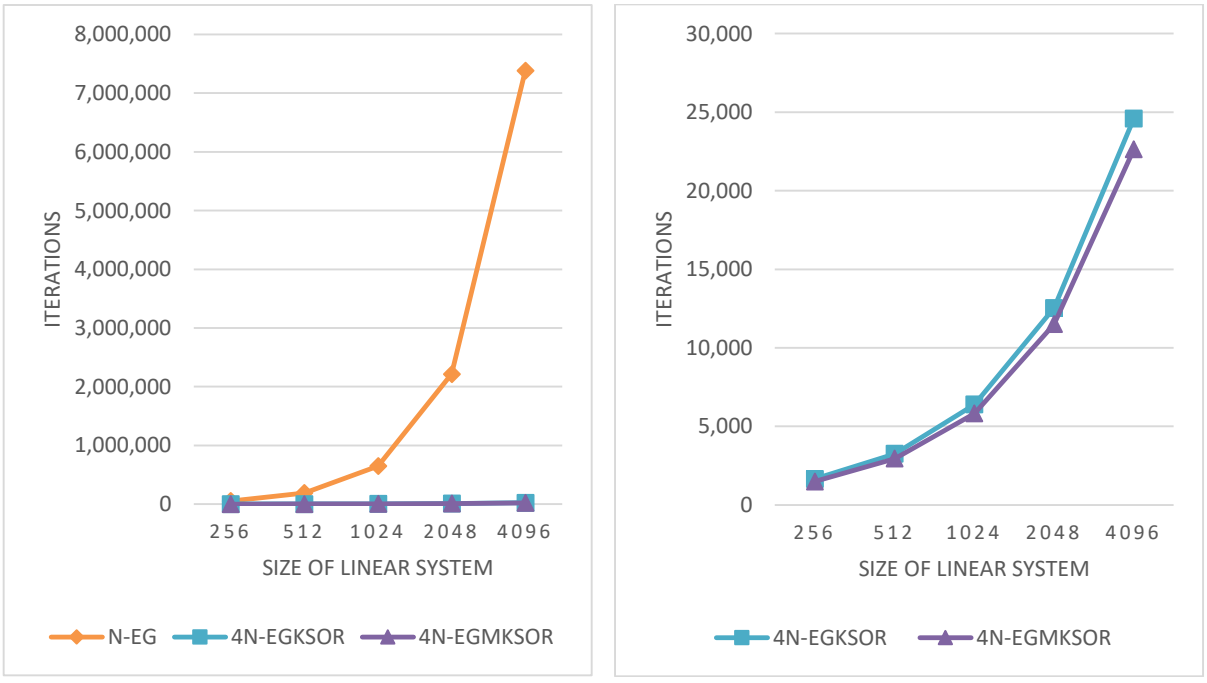


(a) (b)

Fig. 7. (a) Shows the difference in time taken (*seconds*) against the size of linear system for N-EG, 4N-EGKSOR and 4N-EGMKSOR. (b) Shows the close up look of the difference in time taken (*seconds*) against the size of linear system for the 4N-EGKSOR and 4N-EGMKSOR

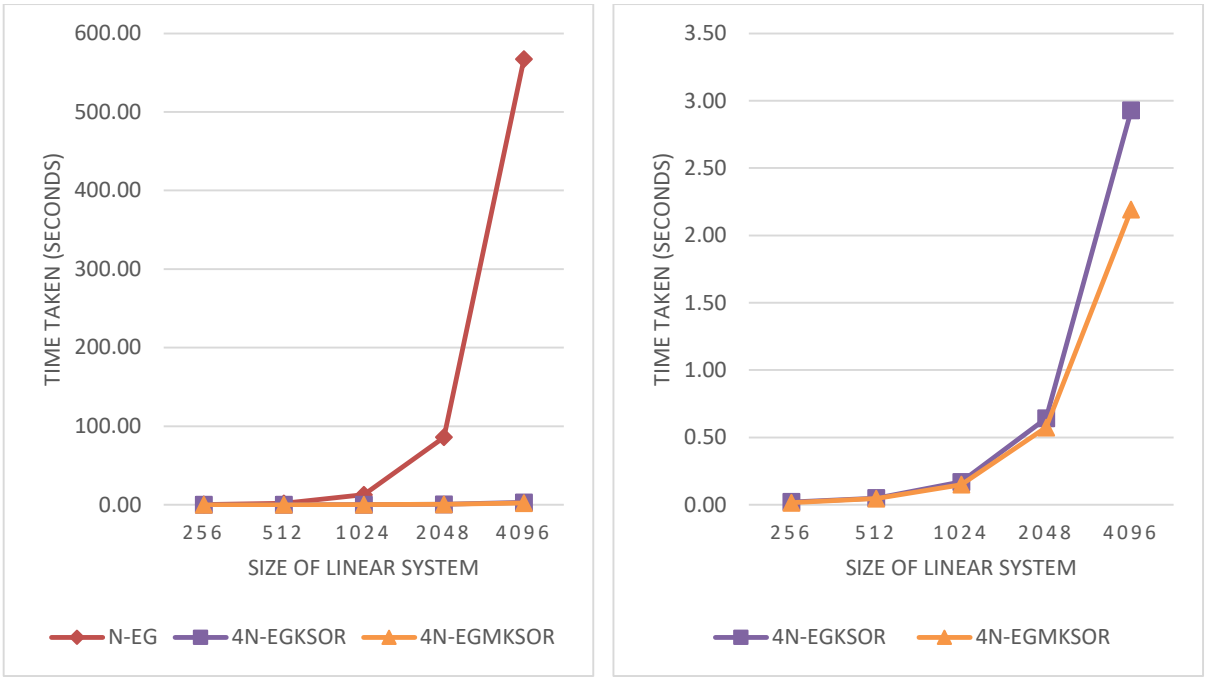
Table 4
The number of iterations, time taken and maximum absolute error generated by N-GS, 4N-EGKSOR and 4N-EGMKSOR at different grid sizes for Problem 4

M	Method	ω_1	ω_2	k	t	E
256	N-EG			54639	0.28	4.5052×10^{-03}
	4N-EGKSOR	-2.0618		1649	0.02	4.5052×10^{-03}
	4N-EGMKSOR	-2.0623	-2.0624	1490	0.02	4.5052×10^{-03}
512	N-EG			189895	1.82	4.5050×10^{-03}
	4N-EGKSOR	-2.0306		3256	0.05	4.5052×10^{-03}
	4N-EGMKSOR	-2.0307	-2.0310	2947	0.05	4.5052×10^{-03}
1024	N-EG			649901	12.70	4.5055×10^{-03}
	4N-EGKSOR	-2.0154		6386	0.17	4.5052×10^{-03}
	4N-EGMKSOR	-2.0155	-2.0157	5820	0.15	4.5052×10^{-03}
2048	N-EG			2217420	85.91	4.5092×10^{-03}
	4N-EGKSOR	-2.0077		12533	0.64	4.5052×10^{-03}
	4N-EGMKSOR	-2.0078	-2.0079	11502	0.57	4.5052×10^{-03}
4096	N-EG			7382332	567.45	4.5241×10^{-03}
	4N-EGKSOR	-2.0039		24584	2.93	4.5052×10^{-03}
	4N-EGMKSOR	-2.0039	-2.0040	22625	2.19	4.5052×10^{-03}



(a) (b)

Fig. 8. (a) Shows the difference in iteration counts against the size of linear system for N-EG, 4N-EGKSOR and 4N-EGMKSOR. (b) Shows the close up look of the difference in iteration counts against the size of linear system for the 4N-EGKSOR and 4N-EGMKSOR



(a) (b)

Fig. 9. (a) Shows the difference in time taken (*seconds*) against the size of linear system for N-EG, 4N-EGKSOR and 4N-EGMKSOR. (b) Shows the close up look of the difference in time taken (*seconds*) against the size of linear system for the 4N-EGKSOR and 4N-EGMKSOR

Table 5
Summary of the reduction in the number of iterations ($\Delta k\%$) and execution time ($\Delta t\%$) of the 4N-EGMKSOR against N-EG for all test problems

M	$\Delta k\%$	$\Delta t\%$
256	97.27 – 97.64	55.88 - 94.62
512	98.43 – 98.66	97.47 - 98.59
1024	99.06 - 99.23	98.79 - 99.05
2048	99.41 - 99.54	99.24 - 99.42
4096	99.61 - 99.72	99.51 - 99.64

5. Conclusion

The 4-Point Newton-Explicit Group Modified Kaudd Successive Over Relaxation (4N-EGMKSOR) iterative method was effectively integrated with the wave variable transformation to numerically solve the porous medium equation with source terms (PMES). The wave variable transformation that reduces the PMES into an ODE with an independent variable ξ has contributed in reducing the computation complexity. Furthermore, the 4-Point Explicit Group iterative strategy also helps facilitate the computational of the linear system as it computes four equations simultaneously. Moreover, the MKSOR iterative method which incorporates two relaxation parameters, which are ω_1 and ω_2 enhance the convergence towards the approximate solution of the PMES. Therefore, the proposed method has been shown to be both efficient and accurate in providing an approximation of the PMES solution. Consequently, the method proposed is a feasible alternative numerical method for estimating solutions to nonlinear parabolic partial differential equations that allow for the existence of travelling wave solutions. The work will extend to a family of half-sweep iterative algorithms [30].

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