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Interval-Valued Fuzzy Bézier Curve Interpolation Model and its Visualization

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ABSTRACT

Curve interpolation is a crucial aspect in several applications; however, traditional approaches often face difficulties in handling data uncertainty and imprecision. This study presents a new interval-valued fuzzy Bézier curve interpolation model designed to address these difficulties effectively. In this paper, interval-valued fuzzy Bézier curve interpolation is introduced. The interval-valued fuzzy control points is defined as the data points to utilizes Bézier blending algorithms to provide smooth and accurate interpolations. In addition, a visualization method is presented to aid in understanding the interpolated curves, improving the interpretation of the results. The model's practical effectiveness is demonstrated through numerical examples. This study enhances curve interpolation techniques by utilizing an interval-valued fuzzy Bézier curve interpolation model. It provides a strong solution for dealing with uncertainty and imprecision in different fields.

1. Introduction

Zadeh [1] proposed fuzzy set theory as a means of dealing with uncertainty. Zadeh [2] later introduced interval-valued fuzzy sets (IVFS), which built upon the concept of fuzzy sets. IVFS utilize interval representations to express membership values as opposed to single values, hence demonstrating greater efficacy in capturing uncertainty when compared to conventional fuzzy sets. Exploration of IVFS has become more varied. Son [3] developed the concept of interval-valued fuzzy soft sets, which provide a flexible representation of membership degrees. Bustince [4] highlighted their importance in the field of soft computing, particularly for specific problem-solving. Furthermore, Verma [5] expanded the concept to hesitant interval-valued fuzzy sets, introducing innovative procedures and features.

These works demonstrate the versatility and effectiveness of IVFS in several fields, such approximate reasoning [6,7], medical diagnosis [8], multivalued logic [9], decision making [10,12], and image processing [13,14]. Given the extensive research and application of IVFS, it is essential to

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integrate them into diverse applications, such as fuzzy geometric modeling. Computer-aided design (CAD) technologies are widely utilized for geometric modeling. The process entails utilizing mathematical representations of geometric elements to depict and manipulate visuals on a computer screen [15]. Geometric modeling is important because it allows for efficient visualization, simplifies design modifications, and enables testing before production, resulting in savings in both cost and time. Bézier techniques are essential because they transform intricate mathematical concepts into a more comprehensible geometric format.

The Bézier curve, a fundamental element of geometric modeling, was first introduced by Pierre Bézier [16]. It was later enhanced by Paul de Casteljau [17] who incorporated Bernstein polynomials in defining curves and surfaces as well as the de Casteljau algorithm. Forrest [18] established the correlation between Bernstein polynomials and Bézier curves. Analyzing Bernstein polynomials from a geometric standpoint entail representing them as functional Bézier curves.

In their study, Wahab *et al.,* [19] introduced a geometric modeling approach that is based on the principles of fuzzy numbers theory. Their concept suggested the use of fuzzy control points to create fuzzy curves and fuzzy surface models in the field of Computer-Aided geometric Design (CAGD). The study investigated the attributes associated with fuzzy control points estimation using fuzzy Bézier, fuzzy B-spline, and fuzzy NURBS techniques. Afterwards, a number of researchers have embraced a comparable structure, merging fuzzy set theories with geometric models, as demonstrated by several authors [20-23].

In order to develop essential theorems, this study aims to analyze and describe the concept of interval-valued fuzzy numbers. The following section will present a clear definition of interval-valued fuzzy control points. The control points will be combined with Bézier basis functions to create the interpolation model for interval-valued fuzzy Bézier curves. The practical usefulness of interval-valued fuzzy Bezier curve interpolation is not well-documented, with only a few real-world applications and case studies available.

1. Preliminaries

This section provides fundamental definitions of interval-valued fuzzy sets, including interval-valued fuzzy numbers, interval-valued fuzzy relations and interval-valued fuzzy points.

Definition 1 [1]. Let A be a fuzzy set defined on a universe X denoted as

$$A = \{\langle x, \mu_A(x) \rangle \mid x \in X\} \tag{1}$$

where $\mu_A(x): X \to [0,1]$ is the membership function. The membership value $\mu_A(x)$ quantifies the extent to which the degree of belongingness of A. Figure 1 shows an example of fuzzy set A.

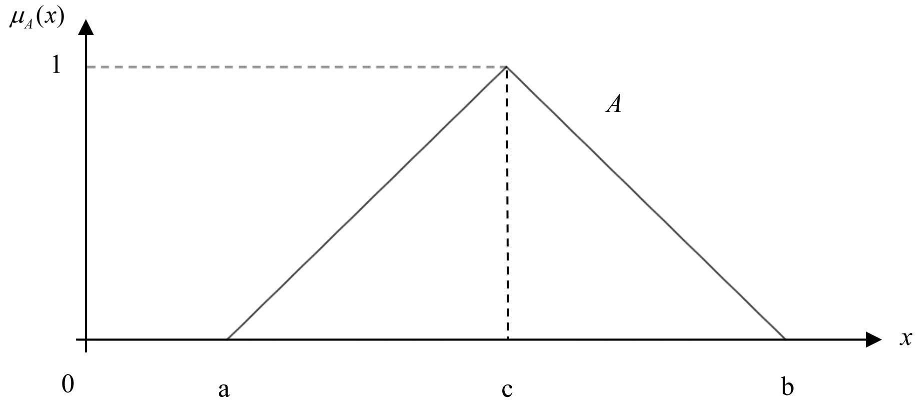

Fig. 1. Fuzzy set *A*

Definition 2 [2]. Let an interval-valued fuzzy set (IVFS) $\tilde{A}$ in X given as

$$\tilde{A} = \left\{ \left\langle x, \underline{\mu}_{\tilde{A}}(x), \overline{\mu}_{\tilde{A}}(x) \right\rangle \mid x \in X \right\} \tag{2}$$

where $\underline{\mu}_{\tilde{A}}(x)$ *and* $\overline{\mu}_{\tilde{A}}(x)$ *are the lower and upper bound of the membership degree respectively on which fulfil the following condition*

$$0 < \underline{\mu}_{\tilde{A}}(x) \le \overline{\mu}_{\tilde{A}}(x) \le 1 \tag{3}$$

The concept of interval-valued fuzzy numbers has emerged from the combination of fuzzy set theory and possibility theory. The concept of fuzzy numbers was first introduced by Gorzałczany [6]. Definition 3 and Definition 4 provides explanations for the interval-valued fuzzy number and the normal triangular interval-valued fuzzy number, respectively.

Definition 3 An interval-valued fuzzy number (IVFN) $\tilde{A}$ is defined as follows:

i. normal where there is any $x_0 \in R$ such that $A(x_0) = [1, 1]$
ii. convex for the memberships $\underline{\mu}_{\tilde{A}}(x)$ and $\overline{\mu}_{\tilde{A}}(x)$ where $\underline{\mu}_{\tilde{A}}(\lambda x_1 + (1-\lambda)x_2) \ge \min \underline{\mu}_{\tilde{A}}(x_1), \underline{\mu}_{\tilde{A}}(x_2)$ and $\overline{\mu}_{\tilde{A}}(\lambda x_1 + (1-\lambda)x_2) \ge \min \overline{\mu}_{\tilde{A}}(x_1), \overline{\mu}_{\tilde{A}}(x_2)$ for $\forall x_1, x_2 \in R, \lambda[0,1]$
iii. $\underline{\mu}_{\tilde{A}}(x)$ and $\overline{\mu}_{\tilde{A}}(x)$ are upper semi-continuous
iv. the support of $\underline{\mu}_{\tilde{A}}(x)$ and $\overline{\mu}_{\tilde{A}}(x)$ are bounded that is the closure of $\left\{ x \in R : \underline{\mu}_{\tilde{A}}(x) > 0 \right\}$ and $\left\{ x \in R : \overline{\mu}_{\tilde{A}}(x) > 0 \right\}$ are bounded.

Definition 4 *According to Chen [24], a normal triangular interval-valued fuzzy number* $\tilde{A}$ *can be represented by two fuzzy numbers* $\underline{A}_x = (\underline{a}_1, \underline{a}_2, \underline{a}_3; \underline{w}_{\tilde{A}})$ *and* $\overline{A}_x = (\overline{a}_1, \overline{a}_2, \overline{a}_3; \overline{w}_{\tilde{A}})$:

$$\tilde{A} = [\underline{A}_x, \overline{A}_x] \tag{4}$$

$$\tilde{A} = [(\underline{a}_1, \underline{a}_2, \underline{a}_3; \underline{w}_{\tilde{A}}), (\overline{a}_1, \overline{a}_2, \overline{a}_3; \overline{w}_{\tilde{A}})] \tag{5}$$

where $\overline{a}_1 \leq \underline{a}_1$, $\underline{a}_3 \leq \overline{a}_3$ and $\underline{w}_{\tilde{A}} = \overline{w}_{\tilde{A}}$ while $\underline{a}_2 = \overline{a}_2$ is a crisp value and $\underline{w}_{\tilde{A}}$ and $\overline{w}_{\tilde{A}}$ are the heights of $\underline{A}_x$ and $\overline{A}_x$. The IVFN can be shown in Figure 2.

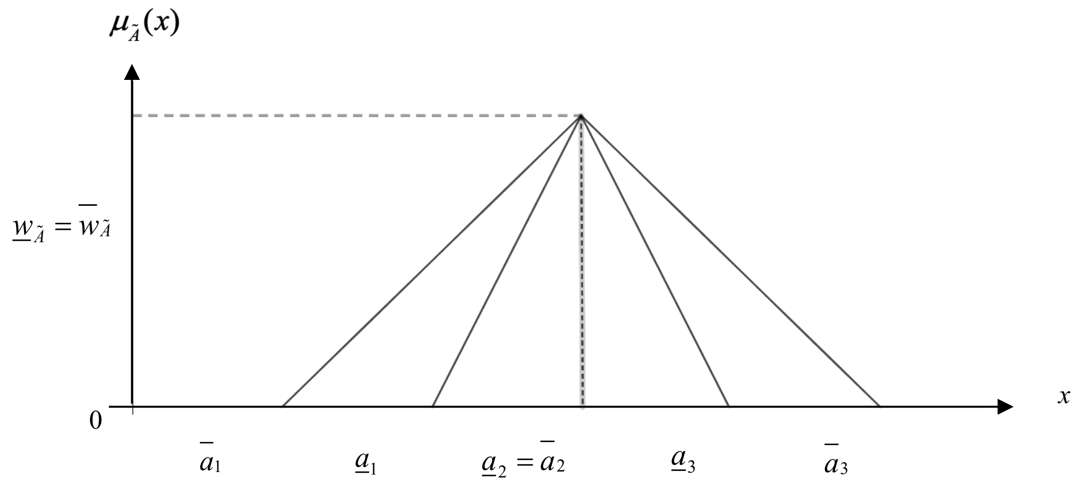

Fig. 2. The normal triangular interval-valued fuzzy number

Fuzzy relation has been examined in [25] and [26]. The interval-valued fuzzy relation (IVFR) is introduced as a transformer that links the definition of the interval-valued fuzzy number (IVFN) to the description of interval-valued fuzzy data points.

Definition 5 [27]. Consider X and Y as non-empty sets and R as interval-valued fuzzy relation. Let $X, Y \subseteq R$ be universal sets, then

$$\tilde{R} = \left\{ \underline{R}(x,y), \overline{R}(x,y) \mid (x,y) \in X \times Y \right\} \tag{6}$$

where $\underline{R} : X \times Y \to [0,1]$ and $\overline{R} : X \times Y \to [0,1]$.

Definition 6 [27]. Let $S, R \in IVFR(X \times Y)$, then for every $(x,y) \in X \times Y$ denoted as

$$S(x,y) \leq R(x,y) \Leftrightarrow \underline{S}(x,y) \leq \underline{R}(x,y) \leq \overline{S}(x,y) \leq \overline{R}(x,y) \tag{7}$$

where there are operations on the supremum and infimum in IVFR $(X \times Y)$ respectively as below

$$(S \vee R)(x,y) = \left[\max\left(\underline{S}(x,y), \underline{R}(x,y)\right), \max\left(\overline{S}(x,y), \overline{R}(x,y)\right) \right] \tag{8}$$

$$(S \wedge R)(x,y) = \left[\min\left(\underline{S}(x,y), \underline{R}(x,y)\right), \min\left(\overline{S}(x,y), \overline{R}(x,y)\right) \right] \tag{9}$$

2. Interval-Valued Fuzzy Bézier Curve Interpolation Model

The data points that are used to determine the shape of a Bézier curve is defined as control point. The control points is represented as $P_i = P_0\ , P_1\ , \ldots, P_i$. The concept of fuzzy control point has been defined in Wahab *et al.,* [19].

2.1 Interval-Valued Fuzzy Bézier Control Point

The interval-valued fuzzy *Bézier* control point (IVFCP) can be expressed using the definitions of interval-valued fuzzy set, interval-valued fuzzy number and interval-valued fuzzy relation as follows.

Definition 8 Let the interval-valued fuzzy set (IVFS) of $\tilde{P}$ in space S such that the set of interval-valued fuzzy control point (IVFCP) is denoted as

$$\tilde{P} = \left\{ \tilde{P}_i \right\}_{i=0}^{n} \tag{10}$$

where there exists membership functions of lower left and right, crisp value, and upper left and right bound respectively $\mu_{\tilde{P}}^{\underline{L}}: S \to [0,1]$, $\mu_{\tilde{P}}^{\bar{L}}: S \to [0,1]$, $\mu_{\tilde{P}}^{C}: S \to [0,1]$, $\mu_{\tilde{P}}^{\underline{R}}: S \to [0,1]$ *and* $\mu_{\tilde{P}}^{\bar{R}}: S \to [0,1]$.

The geometric model integrates the interval-valued fuzzy control point with the basis function of Bézier curve interpolation. The set of interval-valued fuzzy Bézier control points in Eq. (10) includes the lower left, $\tilde{P}_i^{\underline{L}}$, upper left, $\tilde{P}_i^{\bar{L}}$, crisp value, $\tilde{P}^{C}$, lower right, $\tilde{P}_i^{\underline{R}}$, and upper right, $\tilde{P}_i^{\bar{R}}$.

2.2 Interval-Valued Fuzzy Bézier Curve

Definition 9 Consider $\tilde{P}$ be a IVFCP with $\tilde{P} = P_i : i = \{0,1,2,\ldots,n\}$, hence the interval-valued fuzzy Bézier curve (IVFBC) is defined as

$$\tilde{B}(t) = \sum_{i}^{n} \tilde{P}_i J_{n,i}(t), 0 \le t \le 1 \tag{11}$$

where Bernstein polynomials denoted as $J_{n,i}(t) = \binom{n}{i} t^i (1-t)^{n-i}$ and the binomial coefficients are $\binom{n}{i} = \frac{n!}{i!(n-i)!}$.

The interval-valued fuzzy cubic Bézier curve in Eq. (11) is parametric function written as $\tilde{B}(t) = \tilde{P}_0 \tilde{J}_0^n + \tilde{P}_1 \tilde{J}_1^n + \ldots + \tilde{P}_n \tilde{J}_i^n$ which consists of membership curve. The IVFBC consists of

$$\tilde{B}^{\underline{L}}(t) = \sum_{i}^{n} \tilde{P}_i^{\underline{L}} J_{n,i}(t), 0 \le t \le 1 \tag{12}$$

$$\tilde{B}^{\bar{L}}(t) = \sum_{i}^{n} \tilde{P}_i^{\bar{L}} J_{n,i}(t), 0 \le t \le 1 \tag{13}$$

$$\tilde{B}^{C}(t)=\sum_{i}^{n}\tilde{P}_i^{C}J_{n,i}(t),0\le t\le 1 \tag{14}$$

$$\tilde{B}^{\underline{R}}(t)=\sum_{i}^{n}\tilde{P}_i^{\underline{R}}J_{n,i}(t),0\le t\le 1 \tag{15}$$

$$\tilde{B}^{\overline{R}}(t)=\sum_{i}^{n}\tilde{P}_i^{\overline{R}}J_{n,i}(t),0\le t\le 1 \tag{16}$$

where the IVFBC is composed of five components : $\tilde{B}^{\underline{L}}$, $\tilde{B}^{\overline{L}}$, $\tilde{B}^{C}$, $\tilde{B}^{\underline{R}}$ and $\tilde{B}^{\overline{R}}$ which represents as lower left IVFBC, upper left IVFBC, crisp IVFBC, lower right IVFBC and upper right IVFBC respectively.

2.3 Interval-Valued Fuzzy Interpolation Curve

Interval-valued fuzzy interpolation curve can be derived by using the data points from Eq. (12) to Eq. (16) and apply the equation below.

$$\tilde{D}_0=\tilde{J}_0^3(t_0)\tilde{P}_0+\tilde{J}_1^3(t_0)\tilde{P}_1+\tilde{J}_2^3(t_0)\tilde{P}_2+\tilde{J}_3^3(t_0)\tilde{P}_3 \tag{17}$$

$$\tilde{D}_1=\tilde{J}_0^3(t_1)\tilde{P}_0+\tilde{J}_1^3(t_1)\tilde{P}_1+\tilde{J}_2^3(t_1)\tilde{P}_2+\tilde{J}_3^3(t_1)\tilde{P}_3 \tag{18}$$

$$\tilde{D}_2=\tilde{J}_0^3(t_2)\tilde{P}_0+\tilde{J}_1^3(t_2)\tilde{P}_1+\tilde{J}_2^3(t_2)\tilde{P}_2+\tilde{J}_3^3(t_2)\tilde{P}_3 \tag{19}$$

$$\tilde{D}_3=\tilde{J}_0^3(t_3)\tilde{P}_0+\tilde{J}_1^3(t_3)\tilde{P}_1+\tilde{J}_2^3(t_3)\tilde{P}_2+\tilde{J}_3^3(t_3)\tilde{P}_3 \tag{20}$$

There are four points in the Eq. (17) to Eq. (20). To solve this, the matrix equation is used as follows.

$$\begin{bmatrix}\tilde{D}_0\\\tilde{D}_1\\\tilde{D}_2\\\tilde{D}_3\end{bmatrix}=\begin{bmatrix}\tilde{J}_0^3(t_0) & \tilde{J}_1^3(t_0) & \tilde{J}_2^3(t_0) & \tilde{J}_3^3(t_0)\\ \tilde{J}_0^3(t_1) & \tilde{J}_1^3(t_1) & \tilde{J}_2^3(t_1) & \tilde{J}_3^3(t_1)\\ \tilde{J}_0^3(t_2) & \tilde{J}_1^3(t_2) & \tilde{J}_2^3(t_2) & \tilde{J}_3^3(t_2)\\ \tilde{J}_0^3(t_3) & \tilde{J}_1^3(t_3) & \tilde{J}_2^3(t_3) & \tilde{J}_3^3(t_3)\end{bmatrix}\begin{bmatrix}\tilde{P}_0\\\tilde{P}_1\\\tilde{P}_2\\\tilde{P}_3\end{bmatrix} \tag{21}$$

Next, Eq. (21) is simplified into $\tilde{D}=M\tilde{P}$ and lastly into $\tilde{P}=M^{-1}\tilde{D}$.

3. Numerical Example and Algorithm of Interval-Valued Fuzzy Bézier Curve

Table 1 provides a numerical example and algorithm for an Interval-valued Bézier curve with degree $n = 3$.

Table 1
Example of IVFBC with degree 3

IVFBCP/ Membership Degree	0.8	0.5	0.7	0.9
$\tilde{P}_i^{\overline{L}}$	3	8	13	18
$\tilde{P}_i^{\underline{L}}$	5	10	15	20
$\tilde{P}_i^{C}$	8	13	18	23
$\tilde{P}_i^{\underline{R}}$	11	16	21	26
$\tilde{P}_i^{\overline{R}}$	13	18	23	28

Table 1 presents an example of IVFCP with its corresponding degree of membership degree lower left, upper left, crisp value, lower right and upper right. While Figure 3 is the illustrated curve where the IVFCP is blended with the basis function of Bézier in the geometric model as in Eq. (11). Figure 3 displays the lower left IVFBC represented by a yellow curve, next to it is the upper left IVFBC depicted by a blue curve. Thr crisp IVFBC is located at the centre and is coloured green. The lower right IVFBC and upper right IVFBC can be identified by their red and orange curves on the right side.

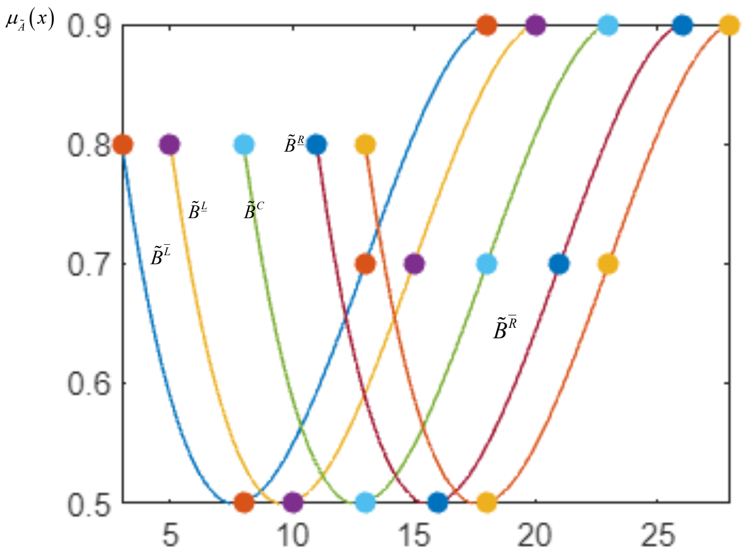

Fig. 3. Interval-valued fuzzy Bézier curve interpolation

The algorithm resulting in IVFBC is as follows:

Step 1 : The interval-valued fuzzy control points (IVFCP) is determined using $\tilde{P} = \left\{\tilde{P}_i\right\}_{i=0}^{n}$.

Step 2 : The interval-valued fuzzy control point is blended with the basis function of Bézier curve interpolation in the geometric model using $\tilde{B}(t)=\sum_{i}^{n}\tilde{P}_i J_{n,i}(t), 0 \le t \le 1$ where the membership functions of lower and upper bound is determined.
Step 3 : The data points or IVFCP of the basis function are transformed into a curve known as IVFBC.

4. Conclusions

This paper presents a novel interpolation model based on interval-valued fuzzy Bézier curves, which efficiently deal with uncertainty present in data. Moreover, the study presents a visualization technique that improves the clarity of the interpolated curves, offering an important understanding of the underlying trends and patterns. Through numerical examples presented in this paper, the model shows the underlying curve within intervals of uncertainty data problem. This research enhances curve interpolation methods by utilizing interval-valued fuzzy data, providing a reliable approach for managing uncertain and imprecise data. In the future, researchers can investigate the specific uses of the suggested model, improve computational methods, and expand the model to include more sources of uncertainty. This will contribute to the progress of curve interpolation and its applications in data analysis, image processing, economic forecasting, engineering design and decision-making.

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