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Journal of Advanced Research in Computing and Applications
ISSN (online): 2462-1927 | Vol. 4, No. 1. Pages 1-8, 2016

Cubic Bezier Homotopy Function for Solving Exponential Equations

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Abstract – *In this study, the new Cubic Bezier homotopy function has been extended from Quadratic Bezier homotopy function for implementation with single exponential equations. The extended function is a combination of the Bezier curve and homotopy function that was based on De Casteljau algorithm. The function will include the target function and the auxiliary function. The target function is the exponential equation while the auxiliary function selected must be controllable and easy to be solved. Then, the extended function has solved by using Newton Homotopy Continuation Method to compute the approximate solutions, accuracy of the approximate solutions and maximum absolute error of the solutions. The analysed results proved that these Cubic Bezier Homotopy function show better approximate solutions, accuracy of the approximate solutions and maximum absolute error of the solutions compared to Quadratic Bezier homotopy function.*

Keywords: Bezier curve, homotopy function, newton homotopy continuation method, single of exponential equation

1.0 INTRODUCTION

Nowadays, a research about concept of Homotopy function has been an interesting and widely used. The majority of them utilized and solve equation comprise of a few classifications, for example, algebraic, polynomials, trigonometric, exponential, and logarithmic equation [1]. In 1930, Lahaye start to introduce the homotopy method [2], but the concept of Homotopy Continuation method for solving equation starts to be identified and studied in 1970. Previous study, Wu have made a lot of study about the Homotopy Continuation method. Wu begin with Modified Chinese algorithm and study the convergence of Newton Homotopy Continuation method. Then, Wu has developed a new formula and compare by using traditional Adomian Decomposition method with Adomian Homotopy Continuation method. Besides that, Wu also solve a nonlinear equation by using Newton Homotopy Continuation method and adjusting an auxiliary function. Then, Wu has modified the secant method with the Homotopy Continuation technique to new Secant Homotopy Continuation function. All the result [3-6] obtained shown that Homotopy Continuation method is better than classical method.

Journal of Advanced Research in Computing and Applications
ISSN (online): 2462-1927 | Vol. 4, No. 1. Pages 1-8, 2016

In [8], Nor also have develop a new function of Quadratic Bezier homotopy function for solving system of exponential equations. Therefore, we extend from the study to develop Cubic Bezier homotopy function for implementation with single exponential equation using Newton Homotopy Continuation method to compute the approximate solutions, accuracy of approximate solutions and maximum absolute error of solutions.

2.0 CUBIC BEZIER HOMOTOPY FUNCTION

Firstly, introduce the Bezier curve based on the De Casteljau algorithm as follow [9],

$$P_i^{(r)} = (1-t)P_i^{r-1}(t) + tP_{i+1}^{r-1}(t) \qquad \begin{cases} r = 1,\dots,k \\ i = 0,\dots,k-r \end{cases} \tag{1}$$

where $P_i^0(t) = P_i$ is the final point in the curve. Then, the Cubic Bezier curve will be construct from the formula in (1) defined by four point P_0, P_1, P_2 and P_3. Then the function become

$$P_0^{(3)}(t) = (1-t)^3 P_0 + 3t(1-t)^2 P_1 + 3t^2(1-t)P_2 + t^2 P_3. \tag{2}$$

The idea of develop Cubic Bezier Homotopy function based on De Casteljau Algorithm. In Homotopy, there is a curve that moving from one curve to another curve. While in De Casteljau, there is movement of point at curve and we consider there is similarity that can relate between them [8]. Therefore, by using method of De Casteljau, it becomes easier and systematic with recursive construction of Cubic Bezier homotopy function is as follow,

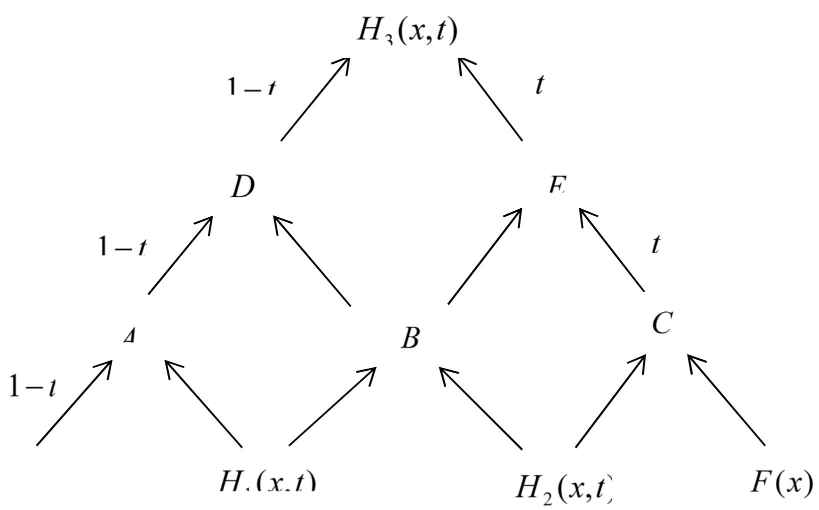

Figure 1: The recursive construction of Cubic Bezier Homotopy function.

Based on the Fig. 1, note that

$$A(x,t) = (1-t)G(x) + tH_1(x,t)$$
$$B(x,t) = (1-t)H_1(x,t) + tH_2(x,t)$$

Journal of Advanced Research in Computing and Applications
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$$C(x,t) = (1-t)H_2(x,t) + tF(x)$$
$$D(x,t) = (1-t)A + tB$$
$$E(x,t) = (1-t)B + tC$$

thus

$$H_3(x,t) = (1-t)D + tE \tag{3}$$

Substitute the above equation,

$$\begin{aligned}
H_3(x,t) &= (1-t)[(1-t)A + tB] + t[(1-t)B + tC] \\
&= (1-t)^2 A + t(1-t)B + t(1-t)B + t^2 C \\
&= (1-t)^2[(1-t)G(x) + tH_1(x,t)] + t(1-t)[(1-t)H_1(x,t) + tH_2(x,t)] \\
&\quad + t(1-t)[(1-t)H_1(x,t) + tH_2(x,t)] + t^2[(1-t)H_2(x,t) + tF(x)] \\
&= (1-t)^3 G(x) + 3t(1-t)^2 H_1(x,t) + t(1-t)^2 H_1(x,t) + t^2(1-t)H_2(x,t) \\
&\quad + t(1-t)^2 H_1(x,t) + t^2(1-t)H_2(x,t) + t^2(1-t)H_2(x,t) + t^3 F(x)
\end{aligned}$$

$$= (1-t)^3 G(x) + 3t(1-t)^2 H_1(x,t) + 3t^2(1-t)H_2(x,t) + t^3 F(x) \tag{4}$$

$$= H_0 B_0^3(t) + H_1 B_1^3 + H_2 B_2^3 + H_3 B_3^3(t) = \sum_{i=0}^{3} H_i B_i^3(t) \tag{5}$$

with

$$\begin{aligned}
&H_0 = G(x) && \text{for } t = 0, \\
&H_1 = H_1(x,t),\ H_2 = H_2(x,t) && \text{for } t = (0,1) \\
&H_3 = F(x) && \text{for } t = 1,
\end{aligned}$$

and $B_i^3(t)$ is a Bernstein function in Bezier curve which is defined in [10]

$$B_i^3(t) = \binom{3}{i}(1-t)^{3-i} t^i = \frac{3!}{i!(3-i)!}(1-t)^{3-i} t^i \tag{6}$$

$$B_0^3(t) = (1)(1-t)^3$$
$$B_1^3(t) = (3)(1-t)^2 t$$
$$B_2^3(t) = (3)(1-t)^2$$
$$B_3^3(t) = t^3$$

where

$i = 0,1,2$

$t \in [0,1]$

Journal of Advanced Research in Computing and Applications
ISSN (online): 2462-1927 I Vol. 4, No. 1. Pages 1-8, 2016

3.0 NEWTON HOMOTOPY CONTINUATION METHOD

The continuation method does not stand alone, we need to combine with other method and then it is called Homotopy Continuation Method (HCM). In this study, the HCM will be combined with the classical method which is Newton methods. Therefore, the new name is Newton Homotopy Continuation method. This method will use to solve the Cubic Bezier homotopy function. Thus, the formula of Newton Homotopy Continuation method is as follow,

$$x_{i+1} = x_i - \frac{H_3(x_i,t)}{D_x H_3(x_i,t)} \quad , \quad i = 1,2,3,\ldots,k \tag{7}$$

where $H_3(x,t)$ is Cubic Bezier homotopy function.

4.0 NUMERICAL EXPERIMENT

Consider the examples of single exponential equation:

Example 1: Consider the following example of single exponential equation that was taken from the previous paper [11]

$$f(x) = x - 2 - e^{-x} \tag{8}$$

The auxiliary function is $g(x) = x - 2$ and the initial value is $x_0 = 2$. Thus, the equation (8) will be implemented into CBHF as follow,

$$\begin{aligned} H_3(x,t) = {} & (1-t)^3(x-2) + 3t(1-t)^2 H_1(x,t) + 3t^2(1-t)H_2(x,t) \\ & + t^3(x-2-e^{-x}) \end{aligned} \tag{9}$$

The comparison results between Quadratic and Cubic Bezier homotopy function are show in Table 1, Table 4 and Table 7 with different numbers of iterations.

Example 2: Consider the following example of single exponential equation that are taken from the previous paper [11]

$$f(x) = 3x^2 - e^{-x} \tag{10}$$

The auxiliary function is $g(x) = x - 0.5$ and the initial value is $x_0 = 0.5$. Thus, the equation (10) will be implemented into CBHF as follow,

$$\begin{aligned} H_3(x,t) = {} & (1-t)^3(x-0.5) + 3t(1-t)^2 H_1(x,t) + 3t^2(1-t)H_2(x,t) + \\ & t^3(3x^2 - e^x) \end{aligned} \tag{11}$$

The comparison results between Quadratic and Cubic Bezier Homotopy function are shows in Table 2, Table 5 and Table 8 with the different step sizes of t .

Example 3: Consider the following example of single exponential equation that are taken from the previous paper [12],

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Journal of Advanced Research in Computing and Applications
ISSN (online): 2462-1927 | Vol. 4, No. 1. Pages 1-8, 2016

$$f(x) = x^3 + e^x - x + 1 \tag{12}$$

The auxiliary function is $g(x) = x + 1$ and the initial value is $x_0 = -1$. Thus, the equation (12) will be implemented into CBHF as follow,

$$\begin{aligned} H_3(x,t) &= (1-t)^3(x+1) + 3t(1-t)^2 H_1(x,t) + 3t^2(1-t)H_2(x,t) \\ &\quad + t^3(x^3 + e^x - x + 1) \end{aligned} \tag{13}$$

The comparison result between Quadratic and Cubic Bezier homotopy function are shows in Table 3, Table 6 and Table 9 with the different step sizes of t.

Table 1: Comparison of the approximate solutions between Quadratic and Cubic Bezier Homotopy Function for equation (8)

Step size, t	**Quadratic Bezier homotopy function, $H_2(x,t)$**	**Cubic Bezier homotopy function, $H_3(x,t)$**
0.1	2.1200275550	2.1200281707
0.01	2.1200282389	2.1200282390
0.001	2.1200282390	2.1200282390

Root of an equation, $x = 2.120028238987641$

Table 2: Comparison of the approximate solutions between Quadratic and Cubic Bezier Homotopy Function for equation (10)

Step size, t	**Quadratic Bezier homotopy function, $H_2(x,t)$**	**Cubic Bezier homotopy function, $H_3(x,t)$**
0.1	0.9100164425	0.9100087659
0.01	0.9100075735	0.9100075725
0.001	0.9100075725	0.9100075725

Root of an equation, $x = 0.910007572488709$

Table 3: Comparison of the approximate solutions between Quadratic and Cubic Bezier Homotopy function for equation (12)

Step size, t	**Quadratic Bezier homotopy function, $H_2(x,t)$**	**Cubic Bezier homotopy function, $H_3(x,t)$**
0.1	-1.3807096756	-1.3807063942
0.01	-1.3807058834	-1.3807058830
0.001	-1.3807058829	-1.3807058829

Root of an equation, $x =$ -1.380705882949681

Table 1, Table 2 and Table 3 show the results of the approximate solutions for equation (8), (10) and (12) respectively. At Table 1 the real root of an equation is 0.910007572488709. The value of approximation solutions at QBHF column is approached to root as the step size of t decreased. Moreover, the approximate solutions are closer to root of an equation when we used CBHF. Among these two homotopy functions, the closest approximations solution to root of an equation is CBHF with step sizes t=0.001. The same results are also appeared at Table 2

Journal of Advanced Research in Computing and Applications
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and Table 3 where the approximate solutions show a better improvement when the step size t was getting smaller.

Table 4: Comparison of the accuracy of approximate solution between Quadratic and Cubic Bezier homotopy function for equation (8)

Step sizes, t	Quadratic Bezier homotopy function, $H_2(x,t)$	Cubic Bezier homotopy function, $H_3(x,t)$
0.1	-7.6606×10^{-7}	-7.6527×10^{-8}
0.01	-6.1422×10^{-11}	-1.1188×10^{-13}
0.001	-6.4393×10^{-15}	-2.7756×10^{-17}

Table 5: Comparison of the accuracy of approximate solution between Quadratic and Cubic Bezier homotopy function for equation (10)

Step sizes, t	Quadratic Bezier homotopy function, $H_2(x,t)$	Cubic Bezier homotopy function, $H_3(x,t)$
0.1	2.6395×10^{-5}	3.5511×10^{-6}
0.01	2.9638×10^{-9}	5.4090×10^{-12}
0.001	3.0020×10^{-13}	4.4409×10^{-16}

Table 6: Comparison of the accuracy of approximate solution between Quadratic and Cubic Bezier homotopy function for equation (12)

Step sizes, t	Quadratic Bezier homotopy function, $H_2(x,t)$	Cubic Bezier homotopy function, $H_3(x,t)$
0.1	-1.8851×10^{-5}	-2.5418×10^{-6}
0.01	-2.0929×10^{-9}	-3.8194×10^{-12}
0.001	-2.1139×10^{-13}	-4.4409×10^{-16}

Table 4, Table 5 and Table 6, show the results of the accuracy of approximate solutions for equation (8), (10) and (12) respectively. At Table 4, both homotopy functions show an improvement of the accuracy of approximate solutions when the step sizes of t become smaller. This means, the accuracy of the approximate solutions gets better as the value of t decreased. Also from the table, CBHF shows the most precise value of the accuracy of approximate solutions which are -2.7756×10^{-17}, 4.4409×10^{-16} and -4.4409×10^{-16} compared to QBHF which are -6.4393×10^{-15}, 3.0020×10^{-13}, and -2.1139×10^{-13}. Next, the same results are shown in Table 5 and Table 6 where the value of the accuracy of approximate solutions approached to zero as the step sizes t were getting smaller. All of the table showed the Cubic Bezier Homotopy function are more accurate based on the accuracy of the approximate solution that approached to zero compared Quadratic Bezier homotopy function. Therefore, it shows that CBHF is the more accurate function to solve this kind of problem.

Journal of Advanced Research in Computing and Applications
ISSN (online): 2462-1927 | Vol. 4, No. 1. Pages 1-8, 2016

Table 7: Comparison of the maximum absolute error of solutions between Quadratic and Cubic Bezier homotopy function for equation (8)

	Error of Solution, ℓ	
Step sizes, t	**Quadratic Bezier homotopy function,** $H_2(x,t)$	**Cubic Bezier homotopy function,** $H_3(x,t)$
0.1	6.8397×10^{-7}	6.8326×10^{-8}
0.01	5.4840×10^{-11}	9.9476×10^{-14}
0.001	5.3291×10^{-15}	4.4409×10^{-16}

Table 8: Comparison of the maximum absolute error of solutions between Quadratic and Cubic Bezier homotopy function for equation (10)

	Error of Solutions, ℓ	
Step sizes, t	**Quadratic Bezier homotopy function,** $H_2(x,t)$	**Cubic Bezier homotopy function,** $H_3(x,t)$
0.1	8.8700×10^{-6}	1.1934×10^{-6}
0.01	9.9600×10^{-10}	1.8179×10^{-12}
0.001	1.0092×10^{-13}	1.1102×10^{-16}

Table 9: Comparison of the maximum absolute error of solutions between Quadratic and Cubic Bezier homotopy function for equation (12)

	Error of Solutions, ℓ	
Step sizes, t	**Quadratic Bezier homotopy function,** $H_2(x,t)$	**Cubic Bezier homotopy function,** $H_3(x,t)$
0.1	3.3926×10^{-6}	5.1121×10^{-7}
0.01	4.2107×10^{-10}	7.6872×10^{-13}
0.001	4.2855×10^{-14}	2.2204×10^{-16}

The comparison results for maximum absolute errors at Table 7, Table 8 and 9, are discussed. All of the tables show the maximum absolute error values of solution become smaller as the value step sizes of t decreased. It indicates that the value of maximum absolute errors approached to zero as the step sizes of t getting smaller. Besides, from each table above shows the improvement value of CBHF which are 4.4409×10^{-16}, 1.1102×10^{-16} and 2.2204×10^{-16} compared to QBHF which are 5.3291×10^{-15}, 1.0092×10^{-13} and 4.2855×10^{-14} . The maximum absolute errors of solution got smaller and converged to zero were resulted as a good and reliable method.

5.0 CONCLUSION

This paper has developed the extended of CBHF from QBHF for solving single exponential equation in order to compute the approximate solution, the accuracy of the approximate solution and the maximum absolute error of solutions. From the result obtained, we can see that the extended function give a better, accurate and precise value of solving this problem compared to previous research which is QBHF. Concluded that, when the approximate solution approached the real root of an equation, the accuracy of the approximate solution tends to zero and the maximum absolute getting smaller. Thus, it shows this extended function is better and more accurate than previous function.

REFERENCES

[1] Ismail, W. Zuria Aizan W. "Newton Homotopy Continuation Method for Solving Nonlinear Equations using Mathematica." Journal of Science and Technology 7, no. 1 (2016).

[2] Rafiq, Arif, and Muhammad Awais. "Convergence on the homotopy continuation method." International Journal of Applications Mathematics and Mechanics 4, no. 6 (2008): 62-70.

[3] Wu, Tzong-Mou. "A modified formula of ancient Chinese algorithm by the homotopy continuation technique." Applied mathematics and computation165, no. 1 (2005): 31-35.

[4] Wu, Tzong-Mou. "A study of convergence on the Newton-homotopy continuation method." Applied Mathematics and Computation 168, no. 2 (2005): 1169-1174.

[5] Wu, Tzong-Mou. "A new formula of solving nonlinear equations by Adomian and homotopy methods." Applied mathematics and computation 172, no. 2 (2006): 903-907.

[6] Wu, Tzong-Mou. "Solving the nonlinear equations by the Newton-homotopy continuation method with adjustable auxiliary homotopy function." Applied mathematics and computation 173, no. 1 (2006): 383-388.

[7] Wu, Tzong-Mou. "The secant–homotopy continuation method." Chaos, Solitons & Fractals 32, no. 2 (2007): 888-892.

[8] Nor, Hafizudin Mohamad, Ahmad Izani Md Ismail, and Ahmad Abdul Majid. "Quadratic bezier homotopy function for solving system of polynomial equations." Matematika 29 (2013): 159-171.

[9] Farin, Gerald. Curves and surfaces for computer-aided geometric design: a practical guide. Elsevier, 2014.

[10] Agoston, Max K. Computer graphics and geometric modeling. Vol. 1. London: Springer, 2005.

[11] Chun, Changbum. "A new iterative method for solving nonlinear equations."Applied mathematics and computation 178, no. 2 (2006): 415-422.

[12] Fang, Liang, Tao Chen, Li Tian, Li Sun, and Bin Chen. "A modified Newton-type method with sixth-order convergence for solving nonlinear equations."Procedia Engineering 15 (2011): 3124-3128.