Penerbit Akademia Baru

Journal of Advanced Research in Computing and Applications
ISSN (online): 2462-1927 | Vol. 3, No. 1. Pages 28-36, 2016

A Scheme Associated with Newton-Cotes of Order One for Solving the Linear Goursat Problem Involving Derivative Term

N. S. Saharizan and M. A. Salim Nasir[a]*

Faculty of Computer and Mathematical Sciences, Universiti Teknologi MARA, 40450 Shah Alam, Selangor, Malaysia.
**syazanasaharizan@yahoo.com, [a]masn@tmsk.uitm.edu.my*

Abstract – *The Goursat partial differential equation is a second order hyperbolic equation that arises in several fields of study. Various numerical methods have been investigated by researchers for solving the Goursat problem such as Adomian Decomposition method, Sumudu Decomposition method, Arithmetic Mean scheme, Centroidal Mean Averaging scheme and Heronian Mean scheme. The purpose of this paper is to develop a new scheme associated with Newton-Cotes of order one for solving the derivative linear Goursat problem. The linear operators and Newton-Cotes of order one are utilized in developing the scheme. The approximation of the derivative term in the functional value plays a big role in providing a good numerical solution for the problem. The high level of accuracy obtained from the numerical experiments validates that the scheme is fit in solving the linear Goursat problem involving derivative term. The comparative study also proved that the scheme has provided a good approximate solution by presenting a better result compared to the results from previous studies. Hence, the accuracy level obtained from the numerical experiments validates the efficiency and reliability of the proposed scheme towards the Goursat problem involving derivative term.*

Keywords: Goursat problem, partial differential equation, numerical integration, accuracy, Newton-Cotes, finite difference formula

1.0 INTRODUCTION

The Goursat partial differential equation arises in various areas of applications, for example in science, physics and engineering. Several researchers have been introduced the Goursat equation as the basis of mathematical models such as in sonic barrier [1], trajectory generation for the N-trailer problem [2], gas dynamics [3], isotropic plate with a curvilinear hole [4] and van der Waals gas expansion [5].

In recent years, a rapid development in finding the approximation solution of the Goursat partial differential equation witnesses many new numerical methods. Some of the numerical methods such as Heronian Mean scheme [6-7], Centroidal Mean Averaging scheme [8] and the Arithmetic Mean scheme [9] have been discovered to investigate the efficiency of the scheme to the Goursat problem involving derivative term.

The Newton-Cotes is a stabilized numerical approach for solving several classes of Goursat problems without derivative term; homogeneous and inhomogeneous linear problems, homogeneous and inhomogeneous non-linear problems [8]. Hence, the Newton-Cotes formula will exhibits in a similar manner and gives significant contribution in qualitative and quantitative behaviours in the approximate solution when solving the Goursat problem involving derivative term.

In this paper, we developed a new scheme associated with Newton-Cotes of order one and the accuracy of the scheme is analyzed by conducting the numerical experiments for two derivative Goursat problems. The aid from MATLAB 7.11.0 (R2010b) is very beneficial in calculating the exact solution and approximate solution at each grid point as well as the relative errors. The relative error will be used to describe the accuracy of the scheme. The accuracy of the scheme will be analyzed by finding the percentage error between the approximate solution and the exact solution. We would like to achieve the objectives of this study by presenting the development of the scheme, the computational results and the comparative study between the proposed scheme and the existing scheme from literature.

2.0 METHODOLOGY

2.1 Linear Operators

The standard form of the Goursat problem is as follows [9]:

$$\begin{aligned} &u_{xy} = f(x, y, u, u_x, u_y) \\ &u(x,0) = g(x), \quad u(0, y) = h(y) \\ &0 \le x \le a, 0 \le y \le b \end{aligned} \tag{1}$$

Two distinct linear operators involved are:

$$L_x = \frac{\partial}{\partial x} \quad \text{and} \quad L_y = \frac{\partial}{\partial y} \tag{2}$$

Applying the linear operators (2) into the general Goursat problem (1) leads to,

$$L_x L_y = f(x, y, u, u_x, u_y) \tag{3}$$

Let the inverse of the operators (2) defined as follows:

$$L_x^{-1}(\cdot) = \int_0^x (\cdot)\, dx \quad \text{and} \quad L_y^{-1}(\cdot) = \int_0^y (\cdot)\, dy \tag{4}$$

The contribution of inverse operators L_y^{-1} to the both sides of (3) yields,

$$L_x[L_y^{-1} L_y u(x, y)] = L_y^{-1} f(x, y, u, u_x, u_y) \tag{5}$$

Or equivalently,

$$L_x[u(x, y) - u(x,0)] = L_y^{-1} f(x, y, u, u_x, u_y) \tag{6}$$

Simplifying (6) gives,

Journal of Advanced Research in Computing and Applications
ISSN (online): 2462-1927 | Vol. 3, No. 1. Pages 28-36, 2016

$$L_x u(x,y) - L_x u(x,0) = L_y^{-1} f(x,y,u,u_x,u_y) \quad (7)$$

The contribution of inverse operators L_x^{-1} to the both sides of (7) yields,

$$L_x^{-1} L_x u(x,y) = L_x^{-1} L_x u(x,0) + L_x^{-1} L_y^{-1} f(x,y,u,u_x,u_y) \quad (8)$$

The expression (8) is equivalent to,

$$u(x,y) = u(x,0) + u(0,y) - u(0,0) + L_x^{-1} L_y^{-1} f(x,y,u,u_x,u_y) \quad (9)$$

Hence, the expression (9) in integral form can be written as follows:

$$u(x_0+h, y_0+h) = u(x_0+h, y_0) + u(x_0, y_0+h) - u(x_0, y_0) + \int_{x_0}^{x_0+h}\int_{y_0}^{y_0+h} f(x,y,u,u_x,u_y)\, dydx \quad (10)$$

2.2 Numerical Integration

In order to investigate the efficiency of the numerical integration, a scheme associated with a numerical integration technique, Newton-Cotes of order one is introduced for solving the hyperbolic Goursat PDEs. The reliability of the numerical integration technique based on Newton-Cotes of order one in approximating the derivative terms u_x and u_y at several grid points will be investigated. Newton-Cotes of order one will provide less computation in developing the scheme because it only involves two endpoints.

The Newton-Cotes of order one formula is as follows [10]:

$$\int_a^b f(x)dx \approx \frac{h}{2}[f(x_0) + f(x_1)] \quad (11)$$

Where $h=(b-a)$, $x_0 = a$ is the lower limit and $x_1 = b$ is the upper limit. For simplicity, we let,

$$F(x,y) = f(x,y,u,u_x,u_y) \quad (12)$$

Utilizing the Newton-Cotes of order one (11) on the double integral in (10) will leads to,

$$\int_{x_0}^{x_0+h}\int_{y_0}^{y_0+h} F(x,y)\, dydx = \frac{h^2}{4}\begin{bmatrix} F(x_0,y_0) + F(x_0+h,y_0) \\ + F(x_0,y_0+h) + F(x_0+h,y_0+h) \end{bmatrix} \quad (13)$$

Then, substitute (13) into (10),

$$\begin{aligned} u(x_0+h, y_0+h) &= u(x_0+h, y_0) + u(x_0, y_0+h) - u(x_0, y_0) \\ &+ \frac{h^2}{4}[F(x_0,y_0) + F(x_0+h,y_0) + F(x_0,y_0+h) + F(x_0+h,y_0+h)] \end{aligned} \quad (14)$$

Substituting (12) into (14) yields,

Journal of Advanced Research in Computing and Applications
ISSN (online): 2462-1927 I Vol. 3, No. 1. Pages 28-36, 2016

$$u(x_0+h\,,\,y_0+h)=u(x_0+h\,,\,y_0)+u(x_0\,,\,y_0+h)-u(x_0\,,\,y_0) + \frac{h^2}{4}\begin{bmatrix} f\big(x_0, y_0, u(x_0, y_0), u_x(x_0, y_0), u_y(x_0, y_0)\big) \\ + f\big(x_0+h, y_0, u(x_0+h, y_0), u_x(x_0+h, y_0), u_y(x_0+h, y_0)\big) \\ + f\big(x_0, y_0+h, u(x_0, y_0+h), u_x(x_0, y_0+h), u_y(x_0, y_0+h)\big) \\ + f\big(x_0+h, y_0+h, u(x_0+h, y_0+h), u_x(x_0+h, y_0+h), u_y(x_0+h, y_0+h)\big) \end{bmatrix} \quad (15)$$

Hence, the new scheme associated with Newton-Cotes of order one is written in (15).

2.3 Finite Difference Formula

We utilized the forward finite difference formula and backward finite difference formula to approximate the values of u_x at point (x_0, y_0), (x_0+h, y_0) and u_y at point (x_0, y_0), (x_0, y_0+h) respectively. The forward difference formulas and backward difference formulas are stated in expression (16), (17), (18) and (19) as following:

$$\frac{\partial u(x_0, y_0)}{\partial x} = \frac{u(x_0+h, y_0)-u(x_0, y_0)}{h} + \mathrm{O}(h) \quad (16)$$

$$\frac{\partial u(x_0, y_0)}{\partial y} = \frac{u(x_0, y_0+h)-u(x_0, y_0)}{h} + \mathrm{O}(h) \quad (17)$$

$$\frac{\partial u(x_0, y_0)}{\partial x} = \frac{u(x_0, y_0)-u(x_0-h, y_0)}{h} + \mathrm{O}(h) \quad (18)$$

$$\frac{\partial u(x_0, y_0)}{\partial y} = \frac{u(x_0, y_0)-u(x_0, y_0-h)}{h} + \mathrm{O}(h) \quad (19)$$

These quantities are used to solved the derivative terms at several points in the grid as at point (x_0, y_0), (x_0+h, y_0) and (x_0, y_0+h).

2.4 Newton-Cotes of Order One

We introduced the numerical integration technique, Newton-Cotes of order one to approximate the values of u_x at points (x_0, y_0+h). We also approximate u_y at points (x_0+h, y_0). Performing the similar technique used to develop the scheme, we discovered the approximation of the unknown value $u_x(x_0\,,\,y_0+h)$ as follows:

$$u_x(x_0, y_0+h)=u_x(x_0\,,\,y_0) + \frac{h}{2}\begin{bmatrix} f\big(x_0, y_0, u(x_0, y_0), u_x(x_0, y_0), u_y(x_0, y_0)\big) \\ + f\big(x_0, y_0+h, u(x_0, y_0+h), u_x(x_0, y_0+h), u_y(x_0, y_0+h)\big) \end{bmatrix} \quad (20)$$

In a similar manner, following is the approximation of the unknown value $u_y(x_0+h\,,\,y_0)$:

Journal of Advanced Research in Computing and Applications
ISSN (online): 2462-1927 | Vol. 3, No. 1. Pages 28-36, 2016

$$u_y(x_0+h,y_0)=u_y(x_0,y_0) + \frac{h}{2}\begin{bmatrix} f(x_0,y_0,u(x_0,y_0),u_x(x_0,y_0),u_y(x_0,y_0)) \\ +f(x_0+h,y_0,u(x_0+h,y_0),u_x(x_0+h,y_0),u_y(x_0+h,y_0)) \end{bmatrix} \quad (21)$$

These quantities are used to solved the derivative terms at several points in the grid as at point (x_0, y_0+h) and (x_0+h, y_0).

2.5 Taylor Series Expansions

From the scheme (15), the value of the function f evaluated at point (x_0+h, y_0+h) consists of the function of $u(x_0+h, y_0+h)$, $u_x(x_0+h, y_0+h)$ and $u_y(x_0+h, y_0+h)$. These three terms will be approximated by using Taylor series expansion. They are:

$$u(x_0+h,y_0+h)=u(x_0+h,y_0)+u(x_0,y_0+h)-u(x_0,y_0)+h^2u_{xy}(x_0,y_0) \quad (22)$$

$$u_x(x_0+h,y_0+h)=u_x(x_0+h,y_0)+hu_{xy}(x_0+h,y_0) \quad (23)$$

$$u_y(x_0+h,y_0+h)=u_y(x_0,y_0+h)+hu_{xy}(x_0,y_0+h) \quad (24)$$

These quantitities are used to solved the derivative terms at point (x_0+h, y_0+h) in the grid.

2.6 Relative Error

In this study, two principal ways; the relative error and the average relative error are significant in validating the efficiency of the scheme.

Definition 1.0: *Let $u_{i,j}$ be the exact solution and $U_{i,j}$ be the approximate solution. The absolute error in the approximation of $U_{i,j} \approx u_{i,j}$ is defined as $|u_{i,j}-U_{i,j}|$. At every grid point in the the discretized domain, the relative error is defined as the ratio of the absolute error where $i=0,1,2,3,\ldots$ and $j=0,1,2,3,\ldots$*

$$E_{i,j}(h)=\frac{|u_{i,j}-U_{i,j}|}{|u_{i,j}|}$$

Definition 2.0: *The number of subinterval n can be defined by $n=\frac{(b-a)}{h}$ where $(b-a)$ is the range of the domain and h is the step size. The average relative error is the total relative error at each grid points over $(n \text{ x } n)$ subinterval.*

$$\overline{E}(h)=\frac{\sum_{i=0}^{n}\sum_{j=0}^{n}E_{i,j}(h)}{n^2}$$

The Definition 1.0 and Definition 2.0 will be used in computing the relative error and average relative error for the entire grid points.

Journal of Advanced Research in Computing and Applications
ISSN (online): 2462-1927 | Vol. 3, No. 1. Pages 28-36, 2016

3.0 RESULTS AND DISCUSSION

3.1 Linear Derivative Goursat Problem

Consider the following derivative Goursat problems.

Case 1:

$$\begin{aligned} &u_{xy} = \frac{u_x + u_y + u}{3} \\ &u(x,0) = e^x \\ &u(0,y) = e^y \\ &0 \le x \le 2 \qquad 0 \le y \le 2 \end{aligned} \tag{25}$$

The analytical solution for Goursat problem (25) is $u(x,y) = e^{x+y}$.

Case 2:

$$\begin{aligned} &u_{xy} = -1 + y + u_x \\ &u(x,0) = -1 + e^x \\ &u(0,y) = -1 + e^y \\ &0 \le x \le 2 \qquad 0 \le y \le 2 \end{aligned} \tag{26}$$

The analytical solution for Goursat problem (26) is $u(x,y) = -1 - xy + e^{x+y}$. A MATLAB program was developed for the derivative Goursat problem (25) and (26). In the next section, we present the relative errors at various grid points. The processes of finding an approximation solution for the linear derivative Goursat problem using the scheme utilizing the Newton-Cotes of order one appears to be complicated since it involves many computations. Hence, the complications of the processes are reduced with the aid from mathematical software, MATLAB 7.11.0 (R2010b). An algorithm is developed as shown in Figure 1,

Let m be the grid points where $m = \frac{(b-a)}{h}$. Let h be the grid size where h=0.0025, 0.005, 0.025 and 0.05.

Step 1: Find the given initial values of $u(x,0)$ and $u(0,y)$.

Step 2: Find the given exact values of $u(x,0)$ and $u(0,y)$.

Step 3: Compute: $f\left(x_0, y_0, u(x_0, y_0), u_x(x_0, y_0), u_y(x_0, y_0)\right)$.

Step 4: Compute: $f\left(x_0 + h, y_0, u(x_0 + h, y_0), u_x(x_0 + h, y_0), u_y(x_0 + h, y_0)\right)$.

Step 5: Compute: $f\left(x_0, y_0 + h, u(x_0, y_0 + h), u_x(x_0, y_0 + h), u_y(x_0, y_0 + h)\right)$.

Step 6: Compute:
$u(x_0 + h, y_0 + h) = u(x_0 + h, y_0) + u(x_0, y_0 + h) - u(x_0, y_0) + h^2 u_{xy}(x_0, y_0)$.

Step 7: Compute: $u_x(x_0 + h, y_0 + h) = u_x(x_0 + h, y_0) + h u_{xy}(x_0 + h, y_0)$.

Journal of Advanced Research in Computing and Applications
ISSN (online): 2462-1927 I Vol. 3, No. 1. Pages 28-36, 2016

Step 8: Compute: $u_y(x_0+h, y_0+h) = u_x(x_0+h, y_0) + hu_{xy}(x_0, y_0+h)$.

Step 9: Compute the scheme:

$$u(x_0+h, y_0+h) = u(x_0+h, y_0) + u(x_0, y_0+h) - u(x_0, y_0) + \frac{h^2}{4}\begin{bmatrix} f\left(x_0, y_0, u(x_0, y_0), u_x(x_0, y_0), u_y(x_0, y_0)\right) \\ + f\left(x_0+h, y_0, u(x_0+h, y_0), u_x(x_0+h, y_0), u_y(x_0+h, y_0)\right) \\ + f\left(x_0, y_0+h, u(x_0, y_0+h), u_x(x_0, y_0+h), u_y(x_0, y_0+h)\right) \\ + f\left(x_0+h, y_0+h, u(x_0+h, y_0+h), u_x(x_0+h, y_0+h), u_y(x_0+h, y_0+h)\right) \end{bmatrix}$$

Step 10: Calculate the relative error between the approximate solution and the exact solution.

Step 11: Calculate the average relative error at each grid size.

Step 12: Repeat step 1 to 11 for i=1, 2, 3, ..., *m* and *j*=1, 2, 3, ..., *m*.

Figure 1: The algorithm for solving the linear Goursat problem with derivative term.

3.2 Accuracy Analysis

The average relative error for the Goursat problem (25) is shown in Table 1. Table 2 shows the comparative study of the scheme (15) for the Goursat problem (25).

Table 1: Average relative errors of the Goursat problem (25) at several grid sizes.

Step size, *h*	Average relative error
0.0025	1.7785136e-007
0.005	7.1043389e-007
0.025	1.7563690e-005
0.05	6.9241405e-005

Table 2: Comparison of the average relative errors between Scheme (15) and Scheme (6).

Step size, *h*	Scheme (15)	Nasir and Ismail [6] Average error, $\overline{E}(h)$ of Arithmetic scheme
0.005	7.1043389e-007	8.5495225e-004
0.01	2.8339244e-006	1.7101055e-003
0.02	1.1272606e-005	3.4209800e-003
0.025	1.7563690e-005	4.2766835e-003

The average relative error for the Goursat problem (26) is shown in Table 3. Table 4 shows the comparative study of the scheme (15) for the Goursat problem (26).

Table 3: Average relative errors of the Goursat problem (26) at several grid sizes.

Step size, *h*	Average relative error
0.0025	1.8728173e-007
0.005	7.4780052e-007
0.025	1.8423636e-005
0.05	7.2277205e-005

Journal of Advanced Research in Computing and Applications
ISSN (online): 2462-1927 I Vol. 3, No. 1. Pages 28-36, 2016

Table 4: Comparison of the average relative errors between Scheme (15) and Scheme (7).

Step size, *h*	**Scheme (15)**	**Nasir & Ismail [7]**
		Average error, $\overline{E}(h)$ of Arithmetic scheme
0.003	3.3437677e-007	1.002e-003
0.006	1.3341132e-006	2.002e-003
0.03	3.2662299e-005	9.934e-003
0.06	1.2708051e-004	1.967e-002

It is observed that the scheme (15) produces smaller average relative error compare to the schemes (6) and (7).

4.0 CONCLUSION

In this study, we have developed a new scheme associated with Newton-Cotes of order one. The accuracy of the scheme was tested by computing the relative errors and average relative errors between the numerical solutions and exact solutions. A good approximation of the scheme (15) indicates that the scheme provides a good accuracy. For the derivative linear Goursat problem (25) and (26), we have investigated the comparative study with the previous literature to support the accuracy analysis for the schemes. From the numerical results, both schemes are better than the previous scheme in the literature. The comparative results from Table 1 shows that the average relative error for Goursat problem (25) at $h = 0.005$ is 7.1043389e-007 compared to the average relative error for the Aritmethic scheme that is 8.5495225e-004. At Table 2, the average relative error for Goursat problem (26) at $h = 0.003$ is 3.3437677e-007 and average relative error for the Aritmnetic scheme is 1.002e-003. Therefore, the scheme (15) is exceedingly accurate since the numerical solutions for the scheme (15) produces better results than the previous study.

REFERENCES

[1] Morawetz, Cathleen Synge. "The mathematical approach to the sonic barrier." Bulletin (New Series) of the American Mathematical Society 6, no. 2 (1982): 127-145.

[2] Tilbury, Dawn, Richard M. Murray, and S. Shankar Sastry. "Trajectory generation for the N-trailer problem using Goursat normal form." IEEE Transactions on Automatic Control 40, no. 5 (1995): 802-819.

[3] Dai, Zihuan, and Tong Zhang. "Existence of a Global Smooth Solution for a Degenerate Goursat Problem of Gas Dynamics." Archive for rational mechanics and analysis 155, no. 4 (2000): 277-298.

[4] Aseeri, S. A. "Goursat functions for a problem of an isotropic plate with a curvilinear hole." Int. J. Open Problems Compt. Math 1, no. 3 (2008).

[5] Lai, Geng. "On the expansion of a wedge of van der Waals gas into a vacuum." Journal of Differential Equations 259, no. 3 (2015): 1181-1202.

[6] Nasir, M. A. S., and A. I. M. Ismail. "A new finite difference scheme based on heronian mean averaging for the Goursat problem." IASME Transactions 2, no. 1 (2005): 104-109.

Journal of Advanced Research in Computing and Applications
ISSN (online): 2462-1927 | Vol. 3, No. 1. Pages 28-36, 2016

[7] Nasir, M.A.S., and Md Ismail, A.I. "A new finite difference scheme for the Goursat problem." International Association of Mechanical Engineers (IASME) Transactions 2, no. 1 (2005): 98-103.

[8] Deraman, R.F. A New Scheme for Solving Goursat Problem using Newton- Cotes Integration Formula (unpublished master thesis). Universiti Teknologi Mara, Shah Alam, Selangor (2014).

[9] Wazwaz, A.M. (2009). Partial differential equations and solitary waves theory. Beijing: Higher of Education Press and Springer-Verlag Berlin Heidelberg.

[10] Burden R. L. and Fairies J. D. (2011). Numerical analysis. Canada: Cengage Learning.