Penerbit Akademia Baru

Journal of Advanced Research in Applied Sciences and Engineering Technology

Journal homepage: www.akademiabaru.com/araset.html
ISSN: 2462-1943

Advanced Research in Applied Sciences and Engineering Technology

An Extended Method for Fitting the First Order Polarization Tensor to a Spheroid

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ABSTRACT

Polarization Tensor (PT) has been widely used in some of the applications of electric and electromagnetic such as electrical imaging, metal detection and electrosensing fish. Furthermore, in these applications, polarization tensors can capture significant information such as material, shape and orientation about the related objects (medical images in electrical imaging or metallic target in metal detection). Some physical information about the unknown objects can then be characterized from the given first order polarization tensor that representing the object. Therefore, it is beneficial to determine an ellipsoid based on the given first order polarization tensor due to the possible similarities between the ellipsoid and the unknown object. The main objective of this study is to present a method in order to determine the semi axes of the spheroid, which is an ellipsoid with two identical axes. The method is an extension of the previous method which is only applicable to two types of spheroid. Using the rotation of the first order polarization tensor, we will show that this extended method can be used to determine all semi axes for any types of spheroid, depending on the given first order PT. After that, some numerical examples are provided specifically to compare between the first order PT for the spheroid with the given first order PT for verifying the results obtained. It is expected that the computed first order PT for the spheroid will be almost similar to the given first order PT.

Keywords:
Matrices; integral equations; rotation

Received: 28 February 2021	Revised: 28 March 2021	Accepted: 25 April 2021	Published: 14 May 2021

1. Introduction

Polarization Tensor (PT) is widely used in in engineering and science. In real world problems, this concept appears in many areas such as in electrical imaging [1], metal detection [2-5] and also to study electrosensing fish [6-10]. In addition, PT is also practised by engineers for the purpose of detecting and clearing land mines from contaminated area [11]. Generally, Polarization Tensor (PT) is used to describe the disturbance in electric and electromagnetic fields due to the presence of conduction objects. The disturbance caused by the conducting objects can be expressed by an asymptotic formula, where, the dominant term of the formula can be represented in terms of PT called as generalized polarization tensor (GPT), as stated by Ammari and Kang [1]. Since the GPT can

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https://doi.org/10.37934/araset.23.1.817

be obtained based on the information about the conducting objects, the GPT itself can then be used to characterize the objects. In electrical imaging, this knowledge is adapted to improve image reconstruction while in metal detection, the GPT is computed and used to describe some metallic detected objects.

Moreover, in some applications, it is also useful to fit the GPT of an object to an ellipsoid so that both the object and the ellipsoid have the same GPT. In [12], for some known objects, an ellipsoid that has the same first order PT (the simplest form of GPT) has been determined in order to develop an experiment for electrosensing fish in discriminating two different objects that have the same first order PT. Table 1 taken from [12], shows a few objects with their first order PT in matrix form together with the semi axes of the ellipsoids that have the same first order PT.

Table 1
The values of the semi axes *a*, *b* and *c* for an ellipsoid, computed when the first order PT for each object is given at conductivity $k = 10^7$, obtained from [12]

Object	**Dimension (cm)**	**Semi axes *a*, *b* and *c* (cm) of the ellipsoid** $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 = 1$	**The first order PT**
Cone	diameter = 3 height = 3	$a = b = 1.28$, $c = 1.40$	$10^{-4}\begin{bmatrix} 0.28 & 0 & 0 \\ 0 & 0.28 & 0 \\ 0 & 0 & 0.31 \end{bmatrix}$
Cylinder	diameter = 3 height = 3	$a = b = 1.69$, $c = 1.99$	$10^{-4}\begin{bmatrix} 0.67 & 0 & 0 \\ 0 & 0.67 & 0 \\ 0 & 0 & 0.82 \end{bmatrix}$
Hemisphere	diameter = 3 height = 3	$a = b = 1.50$, $c = 0.82$	$10^{-4}\begin{bmatrix} 0.31 & 0 & 0 \\ 0 & 0.31 & 0 \\ 0 & 0 & 0.15 \end{bmatrix}$
Pyramid	length =3 width = 3 height =3	$a = b = 1.56$, $c = 1.24$	$10^{-4}\begin{bmatrix} 0.42 & 0 & 0 \\ 0 & 0.42 & 0 \\ 0 & 0 & 0.32 \end{bmatrix}$

In the above table from [12], given the first order PT, the semi axes are computed based on the method proposed in [13]. On the other hand, in their study, Lanneau *et al.*, [14] have used the result in [13] to approximate the dimension of the sphere that has the same first order PT with a cube used in their experiment. After that, a few simulations have been performed by [14] to investigate how underwater robot performs electrical imaging in detecting objects.

In this research, an extended method for fitting the first order PT by a spheroid will be developed. In general, this research is interested in studying the first order PT when the electric fields is disturbed due to the presence of a conducting spheroid. For this study, the PT is referred as the PT for spheroid which is an ellipsoid with two semi axes of equal size. Specifically, the study includes the first order PT for prolate and oblate spheroids. Prolate spheroid has a unique axis that is longer than the other equal axes whereas the oblate spheroid has a unique axis that is shorter than the other identical axes.

2. Mathematical Formulation

According to Ammari and Kang [1], method of asymptotic expansion can be used to represent the disturbance on an electric field due to the presence of conductive object, B. The dominant term of the expansion is the terminology called as generalized polarization tensor (GPT). The simplest form of GPT is called as the first order PT and directly can be determined based on only the geometry and the conductivity of B. In this paper, we focus on the first order PT when B is spheroid. The explicit formula of the first order PT for spheroid considered here actually comes from the first order PT for ellipsoid. The simpler explicit formula of the first order PT when B is an ellipsoid is also provided in [1]. Let B be an ellipsoid with semi principal axes a, b and c represented by the general formula $(\frac{x^2}{a^2})^2 + (\frac{y^2}{b^2})^2 + (\frac{z^2}{c^2})^2 = 1$ in the Cartesian coordinates system where $a, b, c > 0$. In [15], the first order PT for B with conductivity k, can be written as

$$M(k, B) = \begin{bmatrix} M_1 & 0 & 0 \\ 0 & M_2 & 0 \\ 0 & 0 & M_3 \end{bmatrix} \tag{1}$$

where $M_i = (k-1)|B| \frac{1}{(1-d_i)+kd_i}$ for $i = 1,2,3$ and $|B|$ is the volume of B , d_i are the depolarization factors for ellipsoid defined in [16, 17] as

$$d_1(a, b, c) = \frac{abc}{2} \int_0^\infty \frac{1}{(a^2+y)^{3/2}\sqrt{(b^2+y)(c^2+y)}} dy, \tag{2}$$

$$d_2(a, b, c) = \frac{abc}{2} \int_0^\infty \frac{1}{(b^2+y)^{3/2}\sqrt{(a^2+y)(c^2+y)}} dy, \tag{3}$$

$$d_3(a, b, c) = \frac{abc}{2} \int_0^\infty \frac{1}{(c^2+y)^{3/2}\sqrt{(a^2+y)(b^2+y)}} dy, \tag{4}$$

The following propositions which can be found in [16] are the properties of depolarization factors for ellipsoid.

Proposition 1 Depolarization factors (d_1, d_2, d_3) for ellipsoid $(\frac{x^2}{a^2})^2 + (\frac{y^2}{b^2})^2 + (\frac{z^2}{c^2})^2 = 1$ satisfy $d_1 + d_2 + d_3 = 1$.

The integrals in equation (2), (3) and (4) can be further simplified for spheroids, as given in the next proposition.

Proposition 2 Let a, b and c be the semi principal axes of an ellipsoid and d_1 is one of the depolarization factors for the ellipsoid, as defined by (2).

i. If $a > b = c$ then $d_1 = \frac{1-\psi^2}{\psi^2}\left(\frac{1}{2\psi} \ln\left(\frac{1+\psi}{1-\psi}\right) - 1\right)$, where $\psi = \sqrt{1 - \left(\frac{b}{a}\right)^2}$

ii. If $b < a = c$ then $d_1 = \frac{1}{\varphi^2}\left(1 - \frac{\sqrt{1-\varphi^2}}{\varphi} \sin^{-1}\varphi\right)$, where $\varphi = \sqrt{1 - \left(\frac{a}{b}\right)^2}$

The first order PT can be classified as either a positive or negative definite matrix depending on the object conductivity, k and this proven by [1]. Specifically, if B is an ellipsoid, given the first order PT, the conductivity k can then be determined and this is proven by [18]. The combination of these two theoretical results is stated in the next Proposition 3.

Proposition 3 Let M be the first order polarization tensor for an ellipsoid. k >1 if and only if M is a positive definite. Meanwhile, M is negative definite matrix if and only if 0 < k <1.

As spheroid is an ellipsoid with two equal semi axes, equation (1) can then be further simplified. Proposition 4 which are adapted from [19] describes the first order PT for a prolate spheroid whereas the results in Proposition 5 obtained also from [19] summarizes the first order PT for an oblate spheroid.

Proposition 4 Let $M(k, B)$ be the first order PT for spheroid B represented by $(\frac{x^2}{a^2})^2 + (\frac{y^2}{b^2})^2 + (\frac{z^2}{c^2})^2 = 1$ at any conductivity k, where $k > 0$ and $k \neq 1$ while a, b and c are the semi principal axes of B.

i. $a > b = c$ if and only if $M_1 > M_2 = M_3$
ii. $b > a = c$ if and only if $M_2 > M_1 = M_3$
iii. $c > a = b$ if and only if $M_3 > M_1 = M_2$

Proposition 5 Let $M(k, B)$ be the first order PT for spheroid B represented by $(\frac{x^2}{a^2})^2 + (\frac{y^2}{b^2})^2 + (\frac{z^2}{c^2})^2 = 1$ at any conductivity k, where $k > 0$ and $k \neq 1$ while a, b and c are the semi principal axes of B.

i. $a < b = c$ if and only if $M_1 < M_2 = M_3$
ii. $b < a = c$ if and only if $M_2 < M_1 = M_3$
iii. $c < a = b$ if and only if $M_3 < M_1 = M_2$

For this research, we will focus on prolate spheroid with semi axes $a > b = c$, $b > a = c$ and $c > a = b$ as well as oblate spheroid with semi axes $< b = c$, $b < a = c$ and $c < a = b$. Based on Proposition 4 and 5, $M(k, B)$ in (1) for spheroid B with semi axes $a > b = c$ and $b < a = c$ can be reduced to

$$M(k, B) = \begin{bmatrix} M_1 & 0 & 0 \\ 0 & M_2 & 0 \\ 0 & 0 & M_2 \end{bmatrix}. \tag{5}$$

As $d_2 = d_3$ (see [20]), by using Proposition 1,

$$d_2 = \frac{1 - d_1}{2} \tag{6}$$

and we will obtain

$$M_1 = \frac{(k-1)|B|}{(1-d_1)+kd_1}, \tag{7}$$

$$M_2 = \frac{2(k-1)|B|}{1+d_1+k(1-d_1)}. \tag{8}$$

Therefore, $M(k, B)$ for spheroid B with semi axes $a > b = c$ and $a < b = c$ later can be solved analytically using (5) along with (7), (8) and appropriate d_1, given in Proposition 2.

Reversely, if given the first order PT in the form of (5), a spheroid with semi axes $a > b = c$ and $a < b = c$ respectively, depending on either $M_1 > M_2$ or $M_1 < M_2$, can be determined. The method to achieve this can be found in [19], which simplifies the method in [13] for ellipsoid. The algorithm developed in [19] which describes the method on fitting the first order PT to a spheroid with semi axes $a > b = c$ and $a < b = c$ is based on the flow chart in Figure 1.

3. Rotation of the First Order PT for Spheroid

From the flow chart in the previous section, for spheroid $(\frac{x^2}{a^2})^2 + (\frac{y^2}{b^2})^2 + (\frac{z^2}{c^2})^2 = 1$, only prolate spheroid with semi axes $a > b = c$ and oblate spheroid with semi axes $a < b = c$ can be determined from a given first order PT in the form of (5). If the first order PT for the spheroid is in the form

$$M(k,B) = \begin{bmatrix} M_1 & 0 & 0 \\ 0 & M_2 & 0 \\ 0 & 0 & M_1 \end{bmatrix} \tag{9}$$

or

$$M(k,B) = \begin{bmatrix} M_1 & 0 & 0 \\ 0 & M_1 & 0 \\ 0 & 0 & M_3 \end{bmatrix}, \tag{10}$$

a slight modification can be done to the method described in Figure 1 to determine a spheroid with semi axes $b > a = c$ or $c > a = b$ for prolate as well as oblate spheroid with semi axes $b < a = c$ or $c < a = b$. In this case, the result in the next proposition, obtained from [1], will be used.

Proposition 6 Let B and B' be two domains such that $B = RB'$ where R is a unitary transformation and R^T is the transpose of R. Let $M(k,B)$ and $M(k,B')$ be the first order PT associated with B and B' at any k where $0 < k \neq 1 < +\infty$. Then $R^T M(k,B) R = M(k,B')$.

By using the rotation in three dimensions as the unitary matrix transformation, we can say that $M(k,B')$ is the result after $M(k,B)$ is transformed in which $M(k,B)$ is the first order PT for the spheroid B before it is being rotated and $M(k,B')$ is the first order PT for the spheroid B after it is being rotated to B'. The used of R in Proposition 6 can be defined as the following matrices

$$R_y(\theta^o) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \tag{11}$$

$$R_z(\theta^o) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{12}$$

where, they are respectively the usual rotation matrix for θ around the y-axis and z-axis in counter clockwise direction.

Suppose that the first order PT for spheroid at a fixed conductivity k in the form of (9) is given. From Proposition 4 and Proposition 5, we know that the semi axes for the prolate spheroid when

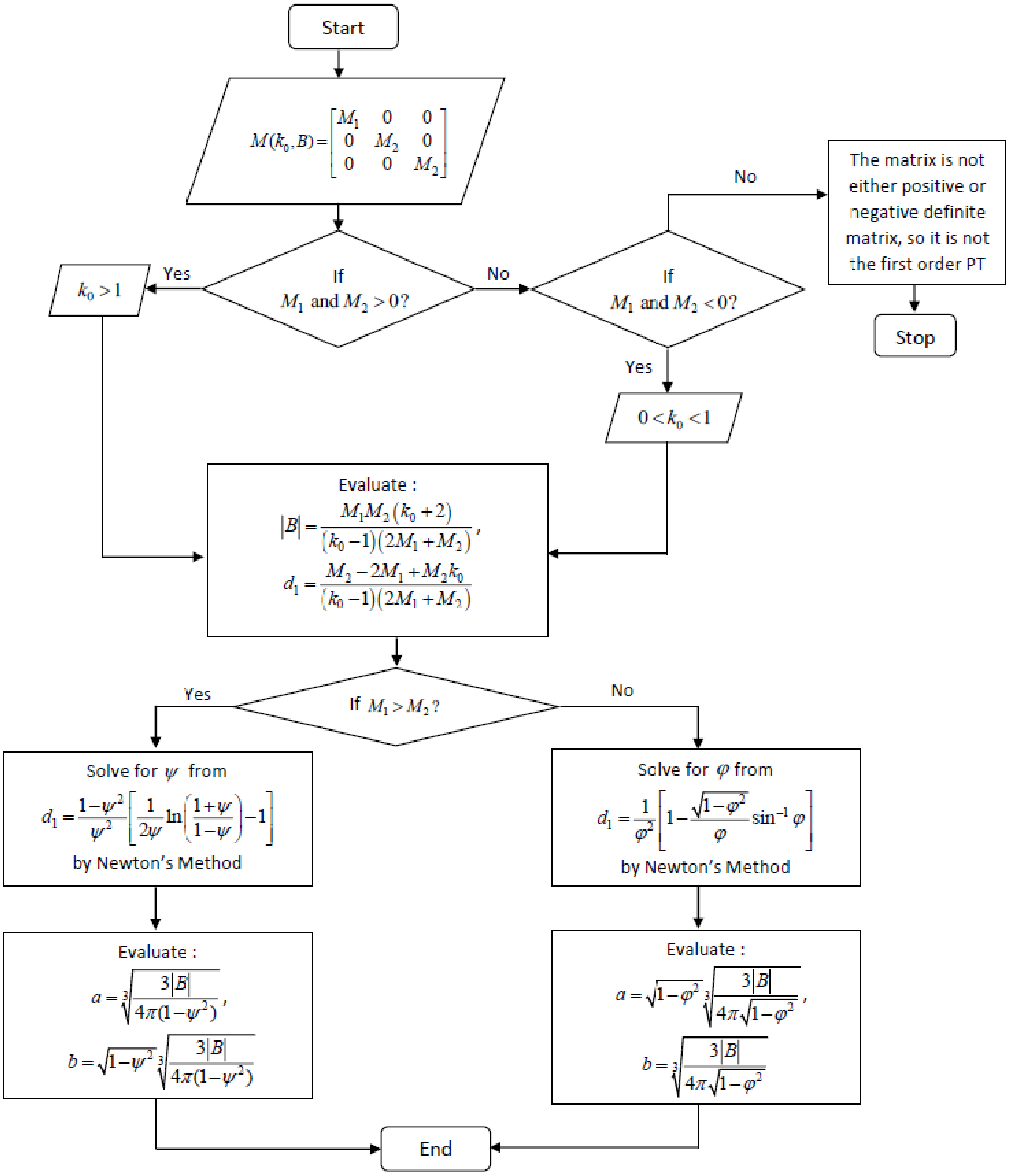

Fig. 1. A flow chart from [19] explaining how to determine the semi axes *a*, *b* and *c* of a spheroid, where, either $a > b = c$ and $a < b = c$ from a given first order PT

$M_2 > M_1$ are $b > a = c$ while when $M_2 < M_1$, the semi axes for the oblate spheroid are $b < a = c$. Geometrically, the spheroid with semi axes $b > a = c$ can be obtained by rotating 90° counterclockwise along the *z*-axis the spheroid with semi axes $a > b = c$. On the other hand, the spheroid with semi axes $b < a = c$ can be obtained by rotating 90° counterclockwise along the *z*-axis the spheroid with semi axes $a < b = c$. Thus, according to Proposition 6, the first order PT for the spheroid with semi axes $b > a = c$ can be obtained using the first order PT for the spheroid with semi axes $a > b = c$ while the first order PT for the spheroid with semi axes $b < a = c$ can be obtained using the first order PT for the spheroid with semi axes $a < b = c$ by the formula

$$\left[R_z(90^\circ)\right]^T M(k,B)R_z(90^\circ) = M(k,B') \tag{13}$$

where, $M(k,B)$ and $M(k,B')$ are respectively the first order PT for the spheroid with semi axes $a > b = c$ and $b > a = c$, or $M(k,B)$ and $M(k,B')$ are respectively the first order PT for the spheroid with semi axes $a < b = c$ and $b < a = c$.

The following are steps proposed in order to find the values of $b > a = c$ or $b < a = c$ from the given first order PT in the form of (9), denoted by M.

1. Change the given first order PT into the form (5). This can be done using the matrix multiplication $R_z(90^\circ)M\left[R_z(90^\circ)\right]^T$, obtained after rearranging (13), since $\left[R_z(90^\circ)\right]^{-1} = \left[R_z(90^\circ)\right]^T$. In this step, the values of M_1 and M_2 in (5) will be replaced respectively with the values M_2 and M_1 from (9).
2. Compute the values for the semi axes *a* and *b* using the method as in Figure 1 which will result $a > b = c$ or $a < b = c$.
3. Interchange the values of *a* and *b* so that $b > a = c$ or $b < a = c$ depending on the values of M_1 and M_2 for the first order PT in the form of (9).

On the other hand, if the first order PT for spheroid at a fixed conductivity *k* in the form of (10) is given, based on Proposition 4 and Proposition 5, the semi axes for the prolate spheroid when $M_3 > M_1$ are $c > a = b$ while when $M_3 > M_1$, the semi axes for the oblate spheroid are $c < a = b$. Geometrically, the spheroid with semi axes $c > a = b$ can be obtained by rotating 90° counterclockwise along the *y*-axis the spheroid with semi axes $a > b = c$ and the spheroid with semi axes $c < a = b$ can be obtained by rotating 90° counterclockwise along the *y*-axis the spheroid with semi axes $a < b = c$. Thus, according to Proposition 6, the first order PT for the spheroid with semi axes $a > b = c$ and $a < b = c$ each can be respectively used to determine the first order PT for the spheroid with semi axes $c > a = b$ or $c < a = b$ from the formula

$$\left[R_y(90^\circ)\right]^T M(k,B)R_y(90^\circ) = M(k,B') \tag{14}$$

and this time, $M(k,B)$ and $M(k,B')$ are respectively the first order PT for the spheroid with semi axes $a > b = c$ and $c > a = b$, or $M(k,B)$ and $M(k,B')$ are respectively the first order PT for the spheroid with semi axes $a < b = c$ and $c < a = b$.

Similarly, the steps to find the values of $c > a = b$ or $c < a = b$ from the given first order PT in the form of (10), denoted by *M*, are proposed as follows:

1. Change the given first order PT into the form (5). This can be done using the matrix multiplication $R_y(90^\circ)M\left[R_y(90^\circ)\right]^T$, obtained after rearranging (14), since $\left[R_y(90^\circ)\right]^{-1} = \left[R_y(90^\circ)\right]^T$. In this step, the values of M_1 and M_2 in (5) will be replaced respectively with the values M_3 and M_1 from (10).
2. Compute the values for the semi axes *a* and *b* using the method as in Figure 1 which will result $a > b = c$ or $a < b = c$.
3. Interchange the values of *a* and c so that $c > a = b$ or $c < a = b$ depending on the values of M_1 and M_3 for the first order PT in the form of (10).

4. Numerical Examples and Discussion

In this section, we will give a few examples on finding the values of the semi axes of a spheroid based on a given first order PT. However, we will only consider the first order PT in the form of (1), that is positive definite, with either $M_3 > M_1 = M_2$ or $M_3 < M_1 = M_2$. Note that the semi axes of the spheroid cannot be directly determined based on the method presented in Figure 1. In this case, the extended method to find the semi axes has been presented in the previous section. Here, we use the first order PT for a few known objects given in Table 1 as examples and determine a spheroid that has the same first order PT with our proposed method for comparison with the spheroid obtained by [12].

Table 2 shows the values for M_1, M_2 and M_3 for the given first order PT as well as the values of all semi axes computed all at conductivity $k_0 = 10^7$, based on the method presented in the previous section for semi axes $c > a = b$ or $c < a = b$. Moreover, the error, *e* in each computation is also included to further support our results. Using the values of *a*, *b* and *c* as presented in the table, the first order PT for the spheroid those semi axes are computed back at $k = 10^7$ based on the original formula (1)-(4). This first order PT is denoted by $\bar{M}$. After letting the given first order PT in the form of (1) equal to $\hat{M}$, we compute the matrix

$$\bar{M} - \hat{M} = \begin{bmatrix} \hat{m}_{11} & 0 & 0 \\ 0 & \hat{m}_{22} & 0 \\ 0 & 0 & \hat{m}_{33} \end{bmatrix} \tag{15}$$

and define the error, *e* to be the matrix norm $e = \sqrt{\hat{m}_{11}{}^2 + \hat{m}_{22}{}^2 + \hat{m}_{33}{}^3}$.

Table 2
The values of the semi axes *a*, *b* and *c* computed at $k_0 = 10^7$ when the first order PT in the form of (1) is given with the listed values for M_1, M_2 and M_3

M_1	M_2	M_3	*a*	*b*	*c*	*e*
0.28×10^{-4}	0.28×10^{-4}	0.31×10^{-4}	1.2841	1.2840	1.3970	3.6002×10^{-8}
0.67×10^{-4}	0.67×10^{-4}	0.82×10^{-4}	1.6870	1.6870	1.9930	8.0546×10^{-8}
0.31×10^{-4}	0.31×10^{-4}	0.15×10^{-4}	1.5040	1.5040	0.8050	4.0359×10^{-8}
0.42×10^{-4}	0.42×10^{-4}	0.32×10^{-4}	1.5610	1.5610	1.2410	4.0991×10^{-8}

As expected, it can be seen in the table that the errors, *e* are very small and equal to 0 at 6 decimal places, meaning that the given first order PT is almost similar to the first order PT for spheroid with the semi axes *a*, *b* and *c*, given in the table. Moreover, the values for the semi axes obtained are similar to the values for the semi axes in Table 1 at 2 decimal places except for *c* in the third row of Table 2 but the value for this *c* is still near to the value of c in Table 1 when $M_1 = M_2 = 0.31 \times 10^{-4}$ and $M_3 = 0.15 \times 10^{-4}$. Thus, our proposed method in this study produce outstanding results in obtaining the semi axes of the spheroid from a given first order PT when compared with the previous literature.

5. Conclusion

In this study, for a spheroid $\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2+\left(\frac{z}{c}\right)^2=1$, an extended method in finding the values of the semi axes for a prolate spheroid with $b>a=c$ and $c>a=b$ as well as for an oblate spheroid with $b<a=c$ and $c<a=b$, from a given first order PT, has been discussed. The method extends the previous method developed specifically to find the values of the semi axes for a spheroid but only for $a>b=c$ and $a<b=c$. By applying the rotation according to Proposition 6, the first order PT for a prolate spheroid with $b>a=c$ and $c>a=b$ can be related to the first order PT for a prolate spheroid with semi axes $a>b=c$ whereas, the first order PT for an oblate spheroid with $b<a=c$ and $c<a=b$ can be related to the first order PT for an oblate spheroid $a<b=c$. Furthermore, using the new proposed methods, numerical examples based on the previous literature are also presented. The findings suggest that the results obtained by the proposed methods have a very strong agreement with the literatures, so the method developed in this research could also be considered by the related researchers in the future.

Acknowledgments
The authors would like to thank Ministry of Education, Malaysia and Research Management Centre (RMC) UTM for providing financial support under vote 5F235 and 5F251 to conduct the study.

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