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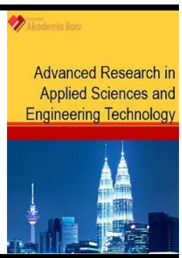

Effects of Variable Thermal Conductivity and Grashof Number on Non-Darcian Natural Convection Flow of Viscoelastic Fluids with Non Linear Radiation and Dissipations

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ABSTRACT

Effects of variable thermal conductivity, porosity and Grashof number on natural convection of viscoelastic fluid flow and heat transfer are studied and discussed. The nonlinear radiation and dissipations are taken into consideration. The fluid flows through non-Darcy porous medium which lies between two heated vertical plates that are kept at constant, but different, temperatures. High order accurate finite difference schemes are introduced to solve the governing equations. The coupled nonlinear differential equations are linearized and iterations are used to approximate that linearized terms. The finite difference method transforms the coupled linearized differential (momentum and energy) equations to a linear system of algebraic equations. An error analysis is introduced by refinement of mesh size and comparisons with available previous results. Effects of fluid and heat parameters on the velocity field, temperature, skin friction factor and Nusselt number are tabulated and plotted. The present results and comparisons show that the numerical solution is of excellent agreement with previous analytical and numerical solution.

Keywords:
Variable conductivity; nonlinear radiation; natural convection; viscoelastic fluid; Forchheimer medium; skin friction; Nusselt number; fourth order finite difference method.

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1. Introduction

The exact and analytical solutions of complicated non-linear coupled differential equations are difficult or impossible to be obtained. Hence, the numerical solutions with high accuracy are very important tools to solve these problems. The non-linearity in momentum equations are raised because of viscoelasticity and non-Darcian effects, while variable conductivity, radiation and dissipation are sources of non-linearity in energy equation. The finite difference method (FDM) is widely used to solve the linear and non-linear differential equations because of simplicity of this method. The natural convection (with variable thermal conductivity) of viscoelastic fluids in Forchheimer medium has many engineering applications such as fiber insulation, heat exchangers,

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nuclear reactors, solar devices, in polymer processing industries, petroleum reservoirs and food industries. The natural convection of non-Newtonian fluids has been studied by many authors [1-11]. Rajagopal and Na [2] introduced a numerical solution for natural convection flow of Rivlin-Ericksen fluid and heat transfer between parallel plates. They studied and computed the skin friction and Nusselt number. Zibabakhsh and Domairy [3] used the homotopy analysis method for solving the natural convection flow of a non-Newtonian fluid between two vertical flat plates. Kargar, and Akbarzade [6] used the homotopy perturbation method (*HPM*) for the study of natural convection flow of a non-Newtonian fluid between two vertical flat plates. Rashidi *et al*., [7] used the differential transformation method (*DTM*) to solve the governing equations of natural convection flow of a third grade non-Newtonian fluids. Murar [9] studied the natural convection flow in a vertical channel in the presence of non-linear radiation and viscous dissipation. He used the finite difference method (*FDM*) to solve the governing coupled equations. Siddiqa *et al*., [10] studied the natural convection flow of a two-phase dusty non-Newtonian power law fluid along a vertical surface. The continuity, momentum and energy equations are solved numerically with the aid of implicit finite difference method (*FDM*). They studied and computed the skin friction and Nusselt number. The natural convection flow in non-Darcy porous media past a vertical surface has been studied by Khani *et al*., [4]. They presented an analytic solution of governing equations of third grade viscoelastic fluid with Darcy-Forchheimer model. Jyoti [11] used the homotopy analysis method (*HAM*) to study the third grade fluid with natural heat convection between two vertical plates.

The nonlinear radiation effect on Newtonian and non-Newtonian fluids has been studied [12-14]. Mushtaq *et al*., [12] introduced a numerical of non-linear radiation heat transfer for the flow of an electrically conducting second grade fluid. Shooting method with fourth and fifth Runge-Kutta integration has been used to solve the governing momentum and energy equations. Ahmed *et al*., [13] introduced a finite element investigation of the flow of a Newtonian fluid in dilating and squeezing porous channel under the influence of non-linear thermal radiation. Ewis [17] introduced a fourth order accurate finite difference method to solve the governing equations of non-linear radiation and dissipation effects on natural convection flow of viscoelastic fluids between vertical plates filled with Forchiemer-Darcy medium. The coupled nonlinear differential equations are linearized and iterations are used to approximate that linearized terms. The finite difference method transforms the coupled linearized differential (momentum and energy) equations to a linear system of algebraic equations. Effects of parameters of fluid and heat on the velocity field, temperature, skin friction factor and Nusselt number have been illustrated and discussed in Figures and Tables. Ewis [18] introduced a second order accurate finite difference method to solve to solve the governing equations of natural convection of non-Newtonian (Rivlin-Ericksen) fluid flow and heat transfer under the influences of non-Darcy resistance force, constant pressure gradient, dissipation and radiation. The novelty of Ewis [10] is to solve this problem between parallel plates channel instead of one plate.

The aim of present work is to study effects of variable thermal conductivity and Grashof number on non-Darcian natural convection flow of viscoelastic fluids between vertical plates. Fourth order accurate finite difference schemes are used to solve the coupled non-linear differential (momentum and energy) equations. A Linearization technique is applied to transform the non-linear terms in momentum and energy equations to linearized ones. The power of the present method of solution is studied by an error analysis which is made via comparisons with available works. Iterations are used up required accuracy. The Accuracy, convergence and stability of present results are satisfied by agreement of present results with available previous works. Skin friction and Nusselt number are computed and compared with the previous works.

2. Basic Equations

Consider a natural convection with thermally variable conductivity and non-Newtonian fluid flows in a Forchheimer medium between two vertically parallel plates as shown in Fig. 1. The two stationary plates have different temperatures T_1 (for left plate) and T_2 (for right plate) with $T_1 > T_2$. The fluid particles, frequently, rise near left plate but they fall near right plate due to their difference in temperatures [2]. The flow is steady and laminar, where viscous dissipation and radiation effects are taken into consideration.

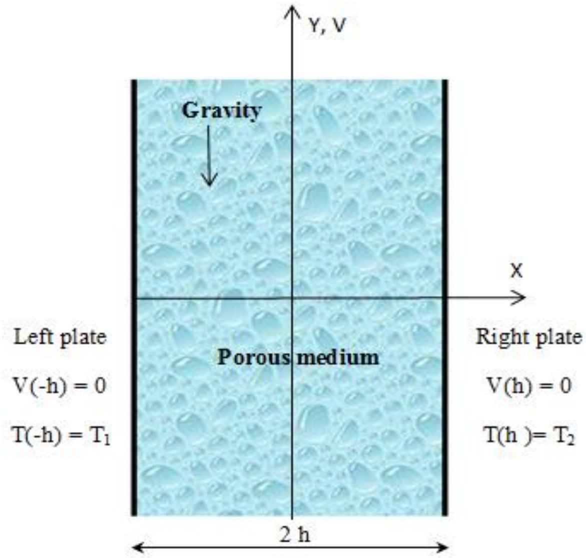

Fig. 1. Channel Geometry and boundary conditions

Thus, the momentum and energy equations are written, respectively, as [2, 4, 14].

$$\mu \frac{d^2V}{dX^2} - \rho_0 U_0 \frac{dV}{dX} + 6\beta_3 \left(\frac{dV}{dX}\right)^2 \frac{d^2V}{dX^2} + \rho_0 \gamma (T - T_m) g - \frac{\mu}{k_p} V - \frac{\rho_0 B}{k_p} V^2 = 0 \tag{1}$$

$$\frac{d}{dX}\left(k \frac{dT}{dX}\right) - \rho_0 C_p U_0 \frac{dT}{dX} + 2\beta_3 \left(\frac{dV}{dX}\right)^4 + \mu \left(\frac{dV}{dX}\right)^2 - \frac{dq_r}{dX} = 0 \tag{2}$$

where,

$$k = k_0 \left[1 + b_1 (T - T_m)\right] \tag{3}$$

$$q_r = -\frac{4\sigma^*}{3k^*}\frac{d}{dX}\left(T^4\right) \tag{4}$$

$$T_m = \frac{T_1+T_2}{2} \tag{5}$$

The boundary conditions are shown in Fig. 1 and they are written as:

$$V(-h) = V(h) = 0 \tag{6}$$

$$T(-h) = T_1, T(h) = T_2\ ,\ -h < X < h \tag{7}$$

To introduce a general solution for any case of dimensions and scales, the following quantities are chosen [2, 4, 6].

$$x = \frac{X}{h},\ v = \frac{V}{U_0},\ \theta = \frac{T-T_m}{T_1-T_2},\ T_r = \frac{T_1}{T_2},\ E_c = \frac{U_0^2}{C_p\,(T_1-T_2)},\ P_r = \frac{\mu\, C_p}{k_o},\ B_r = E_c P_r,\ \delta = \frac{\beta_3 U_0^2}{\mu\, h^2},\ M = \frac{h^2}{k_p},$$
$$R_d = \frac{k_o\, k^*}{4\sigma^* T_1^3},\ S = \frac{\rho_0\, U_0\, h}{\mu},\ F_s = \frac{\rho_0\, B U_0\, h^2}{\mu k_p},\ ,\ .\qquad G_r = \frac{\rho_o\, g\,\gamma\, h^2(T_1-T_2)}{\mu\, U_o},\quad b = b_1(T_1-T_2)$$

Under the above assumptions (Eqns. 4 and 5) and quantities, the dimensionless forms of governing equations (1 and 2) with boundary conditions (6 and 7) are rewritten as

$$\left(1+6\delta\left(\frac{dv}{dx}\right)^2\right)\frac{d^2v}{dx^2} - S\frac{dv}{dx} - M\,v - F_s\, v^2 + G_r\,\theta = 0 \tag{8}$$

$$\frac{d}{dx}\left((1+b\theta)\ \frac{d\theta}{dx}\right) + \frac{4}{3R_d}\frac{d}{dx}\left(\left\{(T_r-1)\theta + \frac{T_r+1}{2}\right\}^3\frac{d\theta}{dx}\right)$$
$$-\ P_r\, S\frac{d\theta}{dx} + B_r\left(2\delta\left(\frac{dv}{dx}\right)^4 + \left(\frac{dv}{dx}\right)^2\right) = 0 \tag{9}$$

$$v(\text{-}1) = v(1) = 0\,,\, \theta(\text{-}1) = \text{-}\,\theta(1) = 0.5 \tag{10}$$

where, $\quad -1 \le x \le 1$.

3. The Numerical Solution

A linearization technique is applied on the system of coupled non-linear ordinary differential equations (8 and 9) as

$$\left(1+6\delta\left(\frac{d\bar{v}}{dx}\right)^2\right)\frac{d^2v}{dx^2} - S\ \frac{dv}{dx} - \left(M + F_s\,\bar{v}\right)v + G_r\,\theta = 0 \tag{11}$$

$$\left[1+b\bar{\theta} + \frac{4}{3R_d}\left((T_r-1)\bar{\theta} + \frac{T_r+1}{2}\right)^3\right]\frac{d^2\theta}{dx^2} + B_r\left[\frac{d\,\bar{v}}{dx} + 2\delta\left(\frac{d\,\bar{v}}{dx}\right)^3\right]\frac{dv}{dx}$$
$$+\left[\left\{b + \frac{4(T_r-1)}{R_d}\left((T_r-1)\,\bar{\theta} + \frac{T_r+1}{2}\right)^2\right\}\frac{d\bar{\theta}}{dx} - P_r\ S\right]\frac{d\theta}{dx} = 0 \tag{12}$$

where, bar notation refers to the iterated (linearized) terms which transform the system (8 and 9) to a linearized one. These linearized equations (11 and 12) with boundary conditions (10) are solved for the flow velocity and temperature using the fourth order finite difference method with an error analysis. The finite domain of solution (-1 < *x* < 1) is divided into *m*-subintervals such that the mesh size is $\Delta = 2/m$, with counter *i*=1, 2, 3, …, *m*+1. The linearized system of coupled non-linear ordinary differential equations (11 and 12) is transformed to system algebraic equations using the fourth order difference schemes. The following fourth order schemes are obtained by Taylor's expansions of the variable $f(x)$ about point. $x_i = (i-1)\Delta - 1$.

$$\frac{df_i}{dx} = \frac{f_{i-2} - 8f_{i-1} + 8f_{i+1} - f_{i+2}}{12\Delta} + O(\Delta^4) \tag{13}$$

$$\frac{d^2 f_i}{dx^2} = \frac{-f_{i-2} + 16f_{i-1} - 30f_i + 16f_{i+1} - f_{i+2}}{12\Delta^2} + O(\Delta^4) \tag{14}$$

$$\frac{df_2}{dx} = \frac{-3f_1 - 10f_2 + 18f_3 - 6f_4 + f_5}{12\Delta} + O(\Delta^4) \tag{15}$$

$$\frac{d^2 f_2}{dx^2} = \frac{10f_1 - 15f_2 - 4f_3 + 14f_4 - 6f_5 + f_6}{12\Delta^2} + O(\Delta^4) \tag{16}$$

$$\frac{df_{n-1}}{dx} = \frac{-f_{n-4} + 6f_{n-3} - 18f_{n-2} + 10f_{n-1} + 3f_n}{12\Delta} + O(\Delta^4) \tag{17}$$

$$\frac{d^2 f_{n-1}}{dx^2} = \frac{f_{n-5} - 6f_{n-4} + 14f_{n-3} - 4f_{n-2} + 15f_{n-1} + 10f_n}{12\Delta^2} + O(\Delta^4) \tag{18}$$

There are two important fluid flow and heat transfer parameters because of their very importance in the engineering applications, since they can be used to improve the shape and efficiency of many equipment in aerodynamics. These quantities are the skin friction factor and Nusselt number factor which are computed after solution the governing equations. The skin friction factor at left plate is defined as [15]

$$\widehat{C}_{f\,L} = \frac{2\,\tau_w}{\rho_0 V_0^2} \tag{19}$$

Nusselt number is defined as the ratio of the convective conduction to the pure molecular thermal conductance [16]. Thus the Nusselt number at left plate may be written as

$$N_{uL} = \frac{4h}{(T_1 - T_2)}\left[1 + \frac{16\sigma^* T_1^3}{3k^* k}\right]\left(-\frac{dT}{dX}\right)\Bigg|_{X=-h} \tag{20}$$

The dimensionless form of these factors are written as

$$C_{f\,L} = \frac{R_e\,\widehat{C}_{f\,L}}{2} = \left[\frac{dv}{dx}\right]_{x=-1} \tag{21}$$

$$N_{uL} = -4\left(1 + \frac{4}{3R_d\{1 + b\theta(-1)\}}\right)\left[\frac{d\theta}{dx}\right]_{x=-1} \tag{22}$$

Fourth order accurate schemes are applied on equations (21 and 22) to minimize round off errors in computations. These schemes can be deduced by Taylor's expansion of independent variables (*v* and *θ*) about *x*=-1. Thus, the dimensionless skin friction factor and Nusselt number of 4th order accurate are rewritten as

$$C_{f\,L} = \frac{-25v_1 + 48v_2 - 36v_3 + 16v_4 - 3v_5}{12\Delta} \tag{23}$$

$$N_{uL} = \left(1+\frac{4}{3R_d\{1+b\theta_1\}}\right)\left(\frac{25\theta_1-48\theta_2+36\theta_3-16\theta_4+3\theta_5}{3\Delta}\right) \tag{24}$$

4. Error Analysis

An error analysis is introduced by refinement of mesh size and comparisons with available previous results. The iterated terms in governing coupled equations need iterations to achieve convergence of the present work. Thus, a good initial guess is required to reach, fast and accurate results. The previous works are used as an initial guess for linearized terms. For number of subintervals m, we find that, the fourth order truncation error of the solution is $O(2/m)^4$. Thus, the *FDM* is a good method to verify the convergence and stability of the analytical and experimental solutions. It is observed that, (5 to 80) iterations are required to achieve (10^{-4} - 10^{-12}) round off error such that number of subintervals ($20 \leq m \leq 2000$).

Table 1
Convergence of present results with relatively small parameters:
$M=F_s=R_d=\delta=1$, $T_r=S=1.25$, $P_r=G_r=0.1$, $E_c=0.2$, $b=1.1$.

	$v(x)$			$\theta(x)$		
m x	20 ($\Delta^4=10^{-4}$)	200 ($\Delta^4=10^{-8}$)	2000 ($\Delta^4=10^{-12}$)	20 ($\Delta^4=10^{-4}$)	200 ($\Delta^4=10^{-8}$)	2000 ($\Delta^4=10^{-12}$)
-1	0	0	0	.5	.5	.5
-0.8	.0027671782	.0027671755	.0027671755	. 4304777493	. 4304780413	. 4304780413
-0.6	.0044796908	.0044796870	.0044796870	. 3571390943	. 3571396927	. 3571396927
-0.4	.0052501400	.0052501345	.0052501345	. 2794144189	. 2794153641	. 2794153640
-0.2	.0052003891	.0052003806	.0052003806	. 1965798522	. 1965811976	. 1965811975
0	.0044727062	.0044726929	.0044726929	. 1076918680	. 1076936882	. 1076936881
0.2	.0032462550	.0032462348	.0032462348	-. 0114813410	. 0114837470	. 0114837469
0.4	.0017616078	.0017615784	.0017615784	-. 0938287143	-. 0938255384	-. 0938255386
0.6	.0003574696	.0003574297	.0003574297	-. 2108620114	-. 2108577077	-. 2108577079
0.8	-.000473674	-.000473724	-.000473724	-. 3437889669	-. 3437826904	-. 3437826906
1	0	0	0	-.5	-.5	-.5

Table 2
Convergence of present results with relatively large parameters:
$M=F_s=\delta=G_r=10$, $P_r=S=5$, $E_c=1$, $Tr=b=1.75$, $R_d=1000$.

	$v(x)$			$\theta(x)$		
m x	20 ($\Delta^4=10^{-4}$)	200 ($\Delta^4=10^{-8}$)	2000 ($\Delta^4=10^{-12}$)	20 ($\Delta^4=10^{-4}$)	200 ($\Delta^4=10^{-8}$)	2000 ($\Delta^4=10^{-12}$)
-1	0	0	0	0.5	0.5	0.5
-0.8	0.0808476124	0.0808453410	0.0808455455	0.5240452081	0.5240417472	0.5240419960
-0.6	0.1514407807	0.1514351189	0.1514356137	0.5385944968	0.5385881399	0.5385886337
-0.4	0.2111394810	0 .2111285421	0.2111294954	0.5465961626	0.5465851797	0.5465859265
-0.2	0.2591880715	0 .2591675687	0.2591693837	0.5503984552	0.5503763405	0.5503773646
0	0.2935965573	0.2935511106	0.2935551578	0.5517455166	0.5516964404	0.5516977299
0.2	0.3010509413	0.3010619700	0.3010772459	0.5528663412	0.5528680038	0.5528690678
0.4	0.2615663918	0.2619808921	0.2620073378	0.5600630284	0.5602722820	0.5602761229
0.6	0.1943157070	0.1949261314	0.1949601189	0.5776426305	0.5807035039	0.5807760365
0.8	0.1057583701	0.1063847465	0.1064216616	0.5571721745	0.5856216484	0.5865494628
1	0	0	0	-0.5	-0.5	-0.5

Tables (3-5) illustrate good agreements of present results with earlier literature works [6,8,11,17]. It is observed that the absolute difference between present results and differential transformation method, *DTM* [17] and homotopy analysis method, *HAM* [11] is less than $3.01*10^{-5}$.

Table 3
Comparison of velocity v with earlier literature works at
δ=0.5, M=F_s=0, R_d=∞, G_r=B_r=1, m=2000

x	Present results ($\Delta^4=10^{-12}$)	HPM [6]	A.E.	RVIM [8]	A.E.	HAM [11]	A.E.
-1	0	0	0	0	0	0	0
-0.8	0.02391934924	0.0239	$1.93*10^{-5}$	0.02356863	$3.51*10^{-4}$	0.02392391	$4.56*10^{-6}$
-0.6	0.03217269108	0.0322	$2.73*10^{-5}$	0.03153540	$6.37*10^{-4}$	0.03217724	$4.55*10^{-6}$
-0.4	0.02840687260	0.0284	$6.87*10^{-6}$	0.02756369	$8.43*10^{-4}$	0.02841114	$4.27*10^{-6}$
-0.2	0.01661764658	0.0166	$1.76*10^{-5}$	0.01565187	$8.43*10^{-4}$	0.01662161	$3.96*10^{-6}$
0	0.00080762874	0.0008	$7.63*10^{-6}$	0.00019888	$6.09*10^{-4}$	0.00081131	$3.68*10^{-6}$
0.2	-0.01508246038	-0.0151	$1.75*10^{-5}$	-0.01604876	$9.66*10^{-4}$	-0.01507910	$3.36*10^{-6}$
0.4	-0.02710371350	-0.0271	$3.71*10^{-6}$	-0.02794788	$8.44*10^{-4}$	-0.0271006	$3.11*10^{-6}$
0.6	-0.03123014822	-0.0312	$3.01*10^{-5}$	-0.03186690	$6.37*10^{-4}$	-0.0312274	$2.75*10^{-6}$
0.8	-0.02342906061	-0.0234	$2.91*10^{-5}$	-0.02378185	$3.53*10^{-4}$	-0.0234270	$2.06*10^{-6}$
1	0	0		0	0	0	0

Table 4
Comparison of temperature θ with earlier literature works at
δ=0.5, M=F_s=0, R_d=∞, G_r=B_r=1, m=2000

x	Present results ($\Delta^4=10^{-12}$)	HPM [6]	A.E.	RVIM [8]	A.E.	HAM [11]	A.E.
-1	0.5	0.5	0	0.49794410	$2.06*10^{-3}$	0.5	0
-0.8	0.4007358824	0.4008	$6.41*10^{-5}$	0.39866980	$2.07*10^{-3}$	0.4007343	$1.58*10^{-6}$
-0.6	0.3011773855	0.3012	$2.26*10^{-5}$	0.29911352	$2.06*10^{-3}$	0.30117607	$1.32*10^{-6}$
-0.4	0.2015909075	0.2016	$9.09*10^{-6}$	0.19953058	$2.06*10^{-3}$	0.20158997	$9.37*10^{-7}$
-0.2	0.1019275017	0.1019	$2.75*10^{-5}$	0.09986820	$2.06*10^{-3}$	0.1019269	$6.02*10^{-7}$
0	0.0020605134	0.0021	$3.95*10^{-6}$	0.00126049	$8.00*10^{-4}$	0.00206022	$2.93*10^{-7}$
0.2	-0.0980700493	-0.0981	$3.00*10^{-5}$	-0.1001317	$2.06*10^{-3}$	-0.09807	$4.93*10^{-8}$
0.4	-0.1984085099	-0.1984	$8.51*10^{-6}$	-0.2004692	$2.06*10^{-3}$	-0.1984082	$3.10*10^{-7}$
0.6	-0.2988285183	-0.2988	$2.85*10^{-5}$	-0.3008857	$2.06*10^{-3}$	-0.2988279	$6.18*10^{-7}$
0.8	-0.3992747317	-0.3993	$2.53*10^{-5}$	-0.4013296	$2.05*10^{-3}$	-0.399274	$7.32*10^{-7}$
1	-0.5	-0.5	0	-0.5	0	-0.5	0

Table 5
Comparison of friction factor C_{fL} with earlier literature works at
M=F_s=1, R_d=100, G_r=B_r=1, m=2000 ($\Delta^4=10^{-12}$)

T_r	⍰=1			⍰=5		
	Present results	Ewis [17]	A.E.	Present results	Ewis [17]	A.E.
1.5	0.1523372982	0.152336641	$6.57*10^{-7}$	0.1359908687	0.135990374	$4.95*10^{-7}$
3.0	0.1667054340	0.166702886	$2.55*10^{-6}$	0.1467168018	0.146714922	$1.88*10^{-6}$
6.0	0.2170205782	0.217015267	$5.31*10^{-6}$	0.1827292493	0.182725509	$3.74*10^{-6}$

Convergence of present solution depending on the influences of v and ϑ by number of sub-intervals (m=20, 200 and 2000) is illustrated in Tables 1 and 2, which give orders of truncation error ($\Delta^4=10^{-4}$, 10^{-8}, 10^{-12}), respectively. Relatively small and large fluid and heat parameters are used to illustrate the power of present method to solve the non-linear differential coupled equations. It is observed that the present solution is stable, convergent and accurate.

5. Results and Discussion

Results are shown in Tables (6-9) and Figures (2-6) to illustrate effects of flow and heat parameters (S, Tr, G_r, b, M, F_s, δ, R_d and B_r) on dimensionless quantities (v, ϑ, C_{fL} and N_{uL}). Certain values of these parameters are chosen to show variation and convergence of present results as they are plotted and tabulated and compared with analytical and numerical available results. The variation of v and ϑ profiles with suction parameter (S=0, 1 and 2) and Grashof number (Gr=10, 20 and 30) are shown in Figs. 2 and 3. It is observed that increasing suction parameter (S) and Grashof number Gr increase v and ϑ. It also is observed that velocity v is relatively affected these parameters more than ϑ. The effect of thermal conductivity (b=0.1, 1 and 2) on the variations of v and ϑ profiles are shown in Fig. 4. It is observed that increasing b increases v and ϑ because of conductivity. It also is observed that velocity v is relatively affected by the thermal conductivity b more than ϑ.

Table 6
Effects of flow and heat parameters on the friction factor C_{fL}
$M=F_s=P_r=1$, $S=0.5$, $Tr=1.5$, $b=1$, $R_d=100$, $m=2000$ ($\Delta^4=10^{-12}$)

G_r	$\delta=1$			$\delta=2$		
	B_r =0.2	0.4	1	0.2	0.4	1
0.5	.1101249834	.1101445049	.1102031831	.1081653491	.1081839007	.1082396616
1	.2089720351	.2091051836	.2095075312	.1985790700	.1986960716	.1990495385
2	.3669229552	.3676844303	.3700295276	.3317063891	.3323203333	.3342087799

Table 7
Effects of flow and heat parameters on the Nusselt number N_{uL} :
$M=F_s=P_r=1$, $S=0.5$, $Tr=1.5$, $b=1$, $R_d=100$, $m=2000$ ($\Delta^4=10^{-12}$)

G_r	$\delta=1$			$\delta=2$		
	B_r =0.2	0.4	1	0.2	0.4	1
0.5	.8846329630	.8832291787	.8790094143	.8846413913	.8832460524	.8790517262
1	.8805769199	.8750975517	.8585322339	.8806774421	.8752993346	.8590423100
2	.8655398373	.8447515749	.7805459745	.8664014240	.8464970055	.7850857079

Table 8
Effects Thermal Conductivity on the friction factor C_{fL}
$M=F_s=P_r=Ec=1$, $S=0.5$, $Tr=1.5$, $R_d=100$, $m=2000$ ($\Delta^4=10^{-12}$)

b	$\delta=1$			$\delta=2$		
	G_r =0.5	1	2	0.5	1	2
0.5	.1042786451	.1991419454	.3544467828	.1025721879	.1897713886	.3210913877
1	.1102031831	.2095075312	.3700295276	.1082396616	.1990495385	.3342087799
1.5	.1150278084	.2178580529	.3824012942	.1128401773	.2064903995	.3445992011

Table 9
Effects of Thermal Conductivity parameters on the Nusselt number N_{uL} :
$M=F_s=P_r=Ec=1$, $S=0.5$, $Tr=1.5$, $R_d=100$, $m=2000$ ($\Delta^4=10^{-12}$)

b	δ=1			δ=2		
	G_r =0.5	1	2	0.5	1	2
0.5	.9995488875	.9787606295	.8991884938	.9995903530	.9792737598	.9039171022
1	.8790094143	.8585322339	.7805459745	.8790517262	.8590423100	.7850857079
1.5	.7904092856	.7703208505	.6941641291	.7904517036	.7708213500	.6985036466

The effect of radiation parameter R_d=0.1, 1 and 2) on the variations of v and ϑ profiles are shown in Fig. 5. It is observed that increasing R_d increases v and ϑ because of heat generation due to radiation. The effect of non-Darcy Forchheimer parameter (F_s=0, 2 and 5) on the variations of v and ϑ profiles is shown in Fig. 6. It is observed that increasing F_s decreases v and ϑ because it represents a resistance to motion. It also is observed that velocity v is relatively affected by F_s more than ϑ.

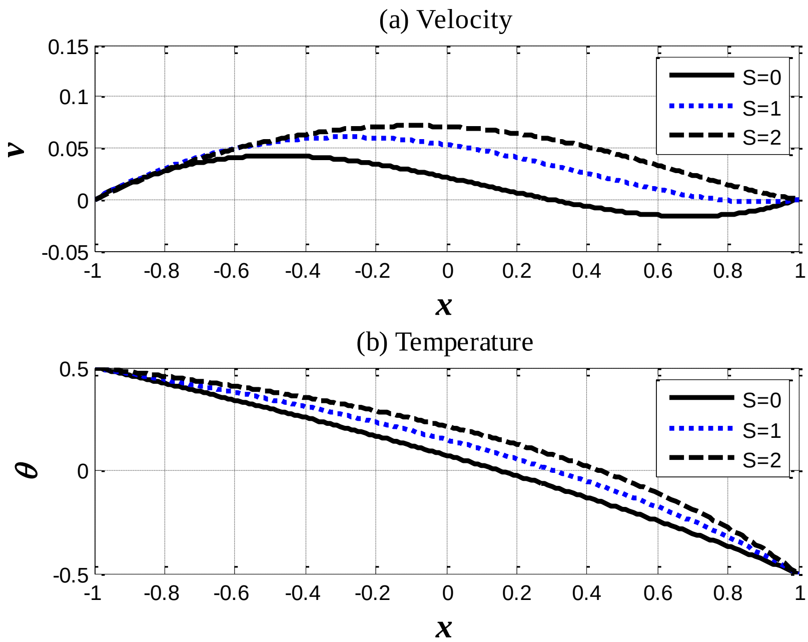

Fig. 2. Variation of v and ϑ profiles with suction parameter (S) when M=1, F_s=1, δ=1, R_d =1 and T_r=1.25, Pr=1, Ec=0.2, Gr=1, b=0.5.

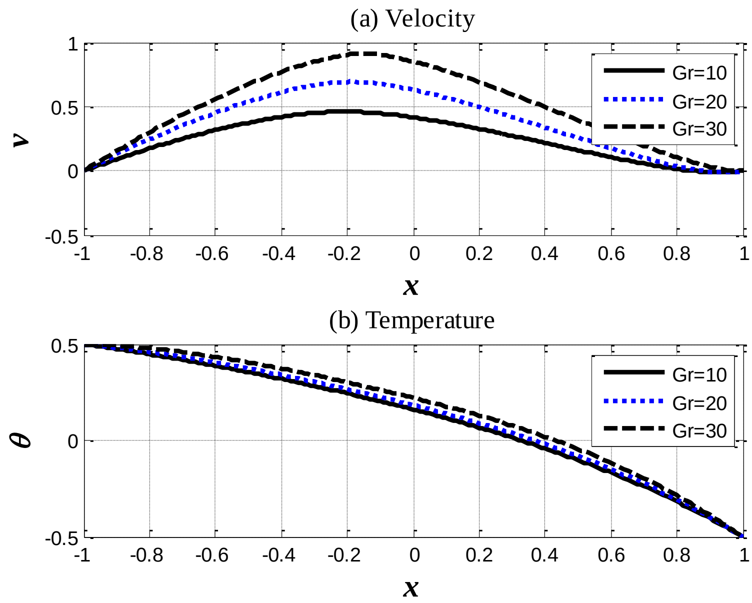

Fig. 3. Variation of v and ϑ profiles with Grashof number (Gr) when M=0, F_s=0, δ=1, R_d =1 and T_r=1.25, Pr=1, S=1, Ec=0.2, b=0.5.

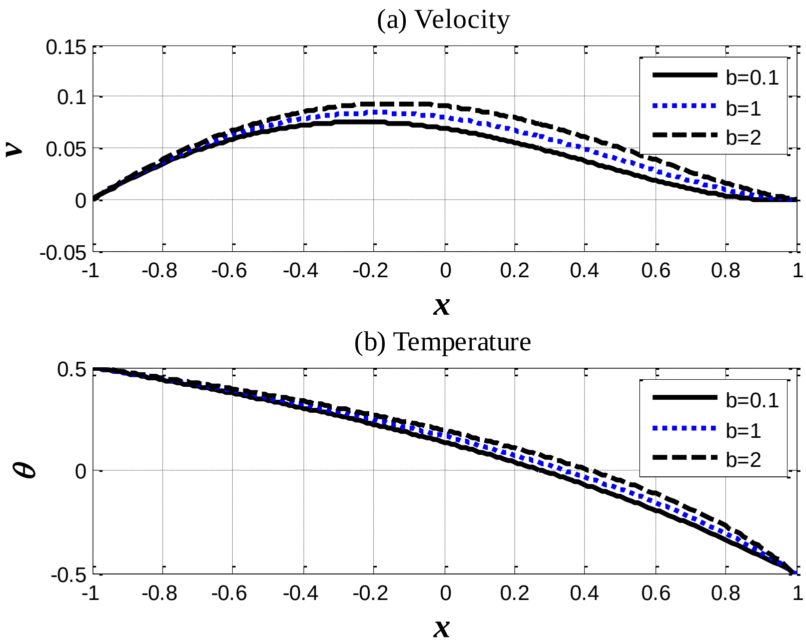

Fig. 4. Variation of v and ϑ profiles with conductivity parameter (b) when M=0, F_s=0, δ=1, R_d =1 and T_r=1.25, Pr=1, S=1, Ec=0.2, Gr=1.

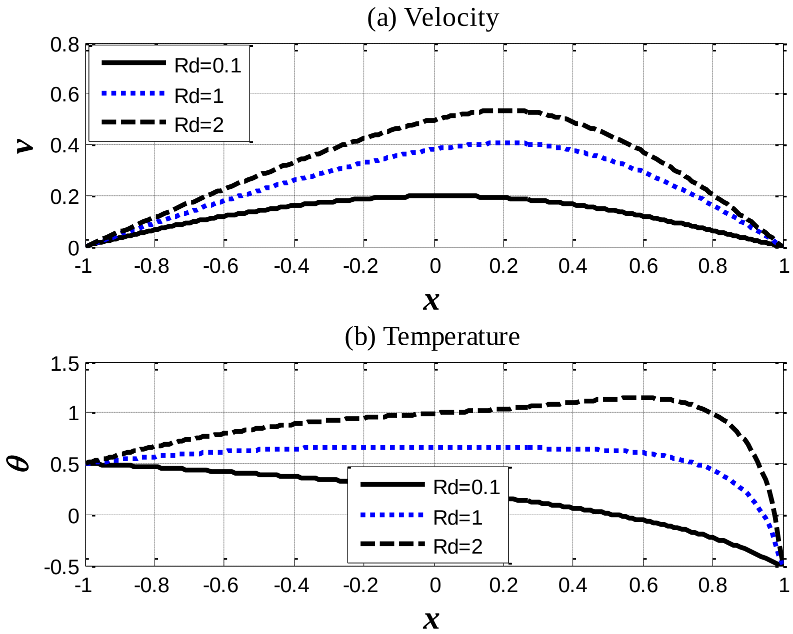

Fig. 5. Variation of v and ϑ profiles with radiation parameter (R_d) when M=1, *Fs=1*, b=1.5, δ=1, T_r=1. 5, Pr=5, S=3, Ec=5, Gr=3.

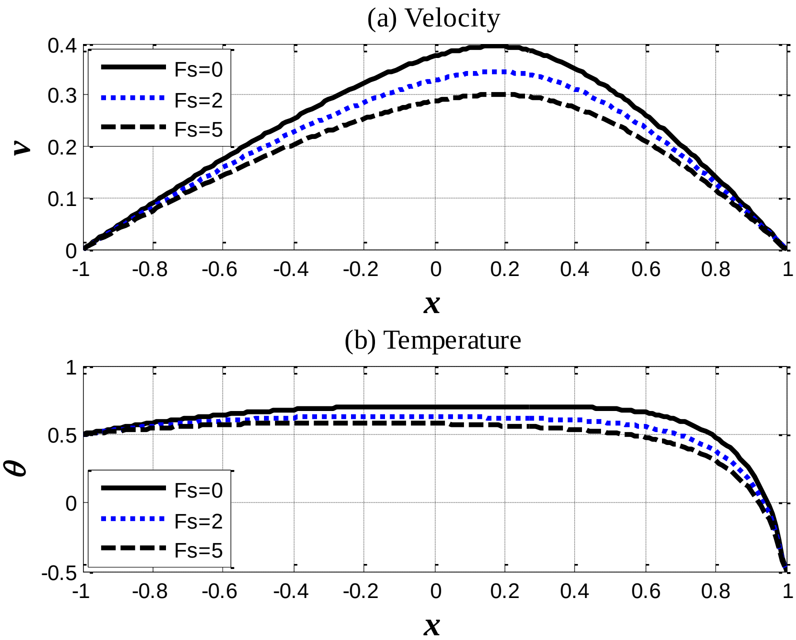

Fig. 6. Variation of v and ϑ profiles with Forchiemer parameter (*Fs*) when M=1, b=1.5, ⍰=2, R_d =1 and T_r=1. 5, Pr=5, S=3, Ec=5, Gr=3.

Tables (6-9) show effects of some fluid flow and heat transfer parameters (B_r, b, G_r and δ) on the friction factor (C_{fL}) and Nusselt number (N_{uL}). It is observed that C_{fL} increases with increasing B_r, b and G_r but, C_{fL} decreases with increasing δ, It is also observed that N_{uL} decreases with increasing B_r, b and G_r but, N_{uL} increases with increasing δ.

6. Conclusion

The variability of thermal conductivity is very important because it represents the real case of energy equation. The effect of variable thermal conductivity on non-Darcian natural convection flow of viscoelastic fluids between vertical plates is studied. The finite difference method with fourth truncation error is used to solve the nonlinear momentum and energy equations between heated vertical plates. The effects of: nonlinear radiation, dissipation and Forchiemer-Darcy resistance force on viscoelastic fluid and heat transfer are taken into consideration. An error analysis is made to achieve accuracy, convergence and stability of present results and their agreement with available previous works. The effects of fluid and heat parameters on velocity, temperature, skin friction factor and Nusselt number are studied and discussed. Some results are listed and shown in tables and figures. It is observed that, increasing Brinkman number and temperature ratio increase velocity, skin friction factor and temperature because of dissipation. It is also observed that, increasing viscoelastic parameter decreases velocity, skin friction factor and temperature because resistance to the flow. The Nusselt number decreases with increasing Brinkman number, but, it increase with increasing both viscoelastic parameter and temperature ratio. It is observed that skin friction factor increases with increasing the thermal conductivity but, skin friction factor. It is also observed that Nusselt number decreases with increasing thermal conductivity. The results which are introduced in tables are very useful in engineering design and comparisons with future analytical and experimental works.

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