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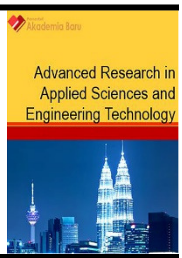

Nonstandard finite difference scheme associated with harmonic mean averaging for the nonlinear Klein-Gordon equation

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ABSTRACT

In this paper, we demonstrate a modified scheme for solving the nonlinear Klein-Gordon equation of PDE hyperbolic types. The Klein-Gordon equation is a relativistic wave equation version of the Schrodinger equation, which is widely used in quantum mechanics. Additionally, the nonstandard finite difference scheme has been used extensively to solve differential equations and we have constructed a modified scheme based on the nonstandard finite difference scheme associated with harmonic mean averaging for solving the nonlinear inhomogeneous Klein-Gordon equation where the denominator is replaced by an unusual function. The numerical results obtained have been compared and showed to have a good agreement with results attained using the standard finite difference (CTCS) procedure, which provided that the proposed scheme is reliable. Numerical experiments are tested to validate the accuracy level of the scheme with the analytical results.

1. Introduction

In the field of physics, the nonlinear Klein-Gordon equation plays an important role especially in the applications of quantum mechanics and condensed matter physics [1,2]. There are many powerful numerical methods that have been applied in order to solve the nonlinear Klein-Gordon equation. The techniques include the finite difference method [3-8], the finite element method [9-11], the inverse scattering radial basis functions (RBF) [12,13], the differential transform method (DYM) [14,15], and the homotopy analysis method (HAM) [16,17].

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Nonstandard finite difference (NSFD) method, which was developed by [22] for some class of differential equations, is the extension of the standard finite difference method and has been used widely in the numerical integration of differential equations. Moreover, [23-25] has identified certain principles for developing the best differential equations using nonlocal approximation by replacing the old denominator of derivatives with a non-negative function, $\phi(h)$ that follows criteria as *h* tends to zero, $\phi(h)$ approaches to zero.

There is a minor study on the nonstandard finite difference method for Klein-Gordon equation. In this paper, we implement the nonstandard finite difference method that is incorporated with harmonic mean averaging to approximate the known function that appears in inhomogeneous Klein-Gordon equation. In [27], by applying the harmonic mean approximation, the results have showed that the numerical and approximated solutions are in good agreement without much loss of accuracy. The harmonic mean (HM) has been stated in [28] as the smallest mean, hence it is suitable to be used in improving the degree of accuracy.

The structure of this paper is organized as follows. In Section 2, we provide with some basic definition of the Klein-Gordon equation and finite difference technique. In Section 3 we apply the proposed method. In Section 4, we present the numerical illustrations for determining the efficiency and reliability of the approach scheme and the conclusion of the study is given in Section 5.

2. Klein-Gordon equation and finite difference technique

2.1. Nonlinear Klein-Gordon equation

Nonlinear Klein-Gordon equation has been studied extensively in science and engineering fields from different perspectives. The general nonlinear Klein-Gordon equation by Wazwaz [29] in the form

$$u_{tt}(x,t) - u_{xx}(x,t) + au(x,t) + F(u(x,t)) = k(x,t) \quad , \quad 0 < x < L \, , \, 0 < t \leq T \tag{1}$$

subject to the initial conditions

$$u(x,0) = f(x) \quad , \quad u_t(x,0) = g(x) \quad , \quad t > 0 \tag{2}$$

where *u* is a function of *x* and *t*, *a* is a constant, $k(x,t)$ is a known function or functional values, $F(u(x,t))$ is a nonlinear function of $u(x,t)$, and $f(x)$ and $g(x)$ are given function.

2.2. Finite difference technique

The formulation of the standard finite difference using Taylor series expansion in central time central space (CTCS) and associate with four points of harmonic mean formula is as follows:

$$\begin{aligned} &U_{i,j+1} + U_{i,j-1} - U_{i+1,j} - U_{i-1,j} + h^2 aU_{i,j} + h^2 F(U_{i,j}) = \\ &h^2 \{[4k_{i,j+1} k_{i+1,j} k_{i,j} k_{i+1,j+1}] / [k_{i+1,j+1} k_{i,j} (k_{i+1,j} + k_{i,j+1}) + k_{i+1,j} k_{i,j+1} (k_{i+1,j+1} + k_{i,j})]\} \end{aligned} \tag{3}$$

where *h* denotes as the grid size. By shifting (i, j) to $(i+1, j+1)$ and then simplifying (3) results in

$$U_{i+2,j+1} = U_{i+1,j+2} + U_{i+1,j} - U_{i,j+1} + h^2 aU_{i+1,j+1} + h^2 F(U_{i+1,j+1}) - h^2 \{[4k_{i,j+1} k_{i+1,j} k_{i,j} k_{i+1,j+1}] / [k_{i+1,j+1} k_{i,j} (k_{i+1,j} + k_{i,j+1}) + k_{i+1,j} k_{i,j+1} (k_{i+1,j+1} + k_{i,j})]\} \quad (4)$$

Therefore, the final form of the general CTCS scheme associated with the harmonic mean averaging can be written as (4). Studies on the use of finite difference schemes which utilize alternatives to other mean averaging method has been reported in [30] for linear Klein-Gordon equation.

3. Nonstandard finite difference harmonic mean scheme

Nonstandard finite difference methods were introduced by Mickens in 1980s [22] as sophisticated numerical techniques, which approximate derivatives and differential equations by using nonlocal discrete representations. In this paper, we analyse the application of a nonstandard finite difference method that is associated with harmonic mean averaging by using $1-e^{-h}$ as the denominator function for nonlinear inhomogeneous Klein-Gordon equation. This denominator function satisfies the property as $h \to 0$, $\phi(h) \to 0$ [22,23]. The final form for our nonstandard finite difference scheme is as follows:

$$U_{i+2,j+1} = U_{i+1,j+2} + U_{i+1,j} - U_{i,j+1} + (1-e^{-h})^2 aU_{i+1,j+1} + (1-e^{-h})^2 F(U_{i+1,j+1}) - (1-e^{-h})^2 \{[4k_{i,j+1} k_{i+1,j} k_{i,j} k_{i+1,j+1}] / [k_{i+1,j+1} k_{i,j} (k_{i+1,j} + k_{i,j+1}) + k_{i+1,j} k_{i,j+1} (k_{i+1,j+1} + k_{i,j})]\} \quad (5)$$

4. Numerical illustrations

To determine the efficiency of the modified scheme described in previous section, we demonstrate some examples.

4.1. Example 1

We first consider the nonlinear inhomogeneous Klein-Gordon equation in [29,30]

$$u_{tt} - u_{xx} + u + u^2 = x^2 \cos^2(t) \quad , \quad 0 < x < 1 \quad , \quad 0 < t < 1 \quad (6)$$

with the following initial conditions:

$$u(x,0) = x \quad , \quad u_t(x,0) = 0 \quad t > 0 \quad (7)$$

The analytical solution of the Example 1 is $u(x,t) = x\cos(t)$ that can be found in [29]. Here, by using scheme (5), we acquire the scheme of Problem 1 below:

$$\begin{aligned} U_{i+2,j+1} = {} & U_{i+1,j+2} + U_{i+1,j} - U_{i,j+1} + (1-e^{-h})^2 U_{i+1,j+1} + (1-e^{-h})^2 (U_{i+1,j+1})^2 - (1-e^{-h})^2 \\ & \{[4 \cdot (x_{i+1})^2 \cos(t_{j+1})^2 \cdot (x_i)^2 \cos(t_j)^2 \cdot (x_{i+1})^2 \cos(t_j)^2 \cdot (x_i)^2 \cos(t_{j+1})^2] / \\ & [((x_{i+1})^2 \cos(t_{j+1})^2 \cdot (x_i)^2 \cos(t_j)^2)((x_{i+1})^2 \cos(t_j)^2 + (x_i)^2 \cos(t_{j+1})^2) + \\ & ((x_{i+1})^2 \cos(t_j)^2 \cdot (x_i)^2 \cos(t_{j+1})^2)((x_{i+1})^2 \cos(t_{j+1})^2 + (x_i)^2 \cos(t_j)^2)]\} \end{aligned} \quad (8)$$

We created computer programs for the application of the standard CTCS scheme and scheme (8) for Example 1. The presented numerical results and graphs in Fig. 1 and Fig. 2 show the respective approximate solution and the relative errors at selected mesh points with several grid sizes.

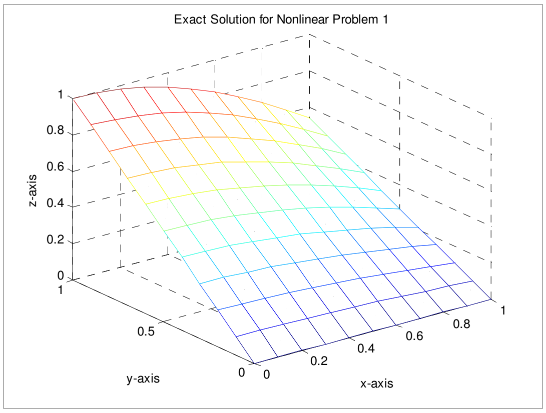

Fig. 1. The Exact Solution of Example 1 in graphical form at h = 0.05

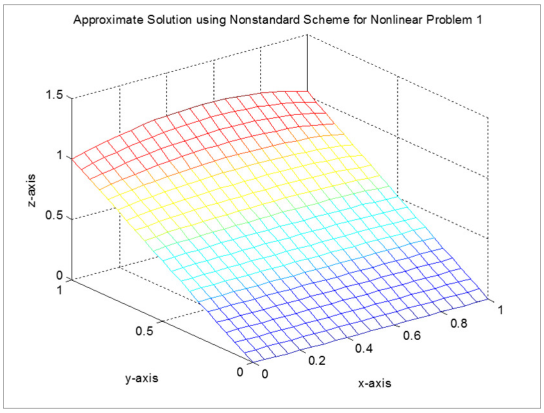

Fig. 2. The Solution of Example 1 in graphical form using scheme (8) at h = 0.05

The above graphical presentations show that the graph for approximate solution in Fig. 2 looks merely the same as the graph for the exact solution in Fig. 1 due to the occurrence of smaller errors.

Table 1
Relative errors for CTCS scheme at selected mesh points with several grid sizes

h	(x, t)			
	(0.25, 0.25)	(0.50, 0.50)	(0.75, 0.75)	(1.00, 1.00)
0.005	0.042963785	0.190844800	0.480046760	0.857451870
0.010	0.042950484	0.190827980	0.481440240	0.864038740
0.025	0.042857395	0.190710250	0.485407250	0.883496240
0.050	0.042525171	0.190290160	0.491298070	0.914870700

Table 2
Relatives errors for scheme (5) at selected mesh points with several grid sizes

h	(x, t)			
	(0.25, 0.25)	(0.50, 0.50)	(0.75, 0.75)	(1.00, 1.00)
0.005	0.042868377	0.190461950	0.479300340	0.857410940
0.010	0.042763824	0.190072460	0.479943580	0.863876860
0.025	0.042420625	0.188895340	0.481644250	0.882517850
0.050	0.041745994	0.186893550	0.483745560	0.911167730

Table 3
Comparison of average relative errors between CTCS scheme and scheme (8)

Scheme	h			
	0.005	0.010	0.025	0.050
CTCS scheme	0.27691523	0.27876396	0.28427012	0.29329484
Scheme (8)	0.27657239	0.27807459	0.28251876	0.28969816

Table 1, Table 2 and Table 3 indicate the relative errors and the average relative errors for the respective CTCS scheme and the approximate scheme (8) for Example 1 at selected grid sizes. The relative errors of the approximate solutions approach zero as the grid size reduces. On the other hand, the average relative errors in Table 3 become smaller when gird size approaches to zero. These evidences indicate both schemes converge.

In addition, as the grid sizes become finer, the numerical approximate solutions converge to the exact solution. Hence, both schemes are consistent and stable as grid sizes tend to zero. However, this illustrates that scheme (5) is more accurate than the CTCS scheme.

4.2. Example 2

We next consider the nonlinear inhomogeneous Klein-Gordon equation in [20,31]

$$u_{tt} - u_{xx} + \frac{\pi^2}{4} u + u^2 = x^2 \sin\left(\frac{\pi^2}{2} t\right) \quad , \quad 0 < x < 1 \quad , \quad 0 < t < 1 \tag{9}$$

with initial conditions

$$u(x,0) = 0 \quad , \quad u_t(x,0) = \frac{\pi}{2} x \qquad t > 0 \tag{10}$$

The analytical solution of Problem 2 is $u(x,t) = x\sin\left(\frac{\pi}{2} t\right)$ that can be found in [20]. Here, by using scheme (5), we obtained the new approximate scheme according to Example 2 as follow:

$$U_{i+2,j+1} =$$

$$U_{i+1,j+2} + U_{i+1,j} - U_{i,j+1} + \left(1-e^{-h}\right)^2 \left(\frac{\pi^2}{4}\right) U_{i+1,j+1} + \left(1-e^{-h}\right)^2 \left(U_{i+1,j+1}\right)^2 - \left(1-e^{-h}\right)^2$$

$$\left\{\left[4 \cdot \left(x_{i+1}\right)^2 \sin\left(\frac{\pi^2}{2} t_{j+1}\right) \cdot \left(x_i\right)^2 \sin\left(\frac{\pi^2}{2} t_j\right) \cdot \left(x_{i+1}\right)^2 \sin\left(\frac{\pi^2}{2} t_j\right) \cdot \left(x_i\right)^2 \sin\left(\frac{\pi^2}{2} t_{j+1}\right)\right] \right/ \tag{11}$$

$$\left[\left(\left(x_{i+1}\right)^2 \sin\left(\frac{\pi^2}{2} t_{j+1}\right) \cdot \left(x_i\right)^2 \sin\left(\frac{\pi^2}{2} t_j\right)\right)\left(\left(x_{i+1}\right)^2 \sin\left(\frac{\pi^2}{2} t_j\right) + \left(x_i\right)^2 \sin\left(\frac{\pi^2}{2} t_{j+1}\right)\right) + \right.$$

$$\left.\left.\left(\left(x_{i+1}\right)^2 \sin\left(\frac{\pi^2}{2} t_j\right) \cdot \left(x_i\right)^2 \sin\left(\frac{\pi^2}{2} t_{j+1}\right)\right)\left(\left(x_{i+1}\right)^2 \sin\left(\frac{\pi^2}{2} t_{j+1}\right) + \left(x_i\right)^2 \sin\left(\frac{\pi^2}{2} t_j\right)\right)\right]\right\}$$

We developed computer programs for the application of CTCS scheme and scheme (11) for Example 2. The presented numerical results and graphs in Fig. 3 and Fig. 4 demonstrate the respective approximate solution and relative errors at selected mesh points with several grid sizes.

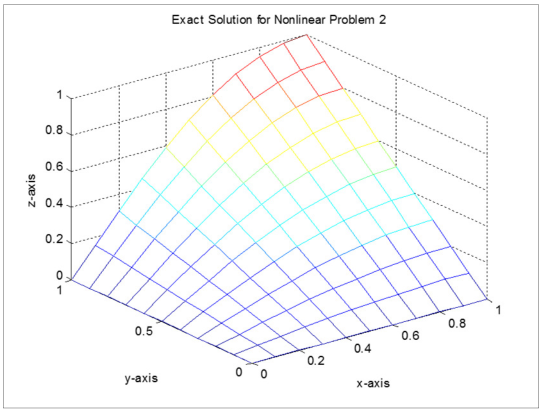

Fig. 3. The Exact solution of Example 2 in graphical form at h = 0.05

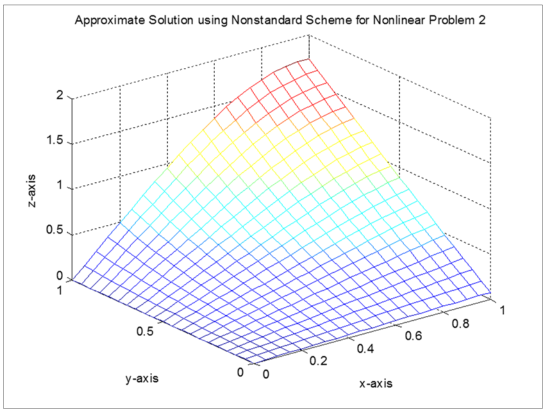

Fig. 4. The Solution of Example 2 in graphical form using scheme (11) at h = 0.05

The above graphical presentations, show that the graph for approximate solution in Fig. 4 looks the same as the graph for the exact solution in Fig. 3 due to the occurrence of smaller errors.

Table 4
Relative errors for CTCS scheme at selected mesh points with several grid sizes

h	*(x, t)*			
	(0.25, 0.25)	(0.50, 0.50)	(0.75, 0.75)	(1.00, 1.00)
0.005	0.074155149	0.243917090	0.501234700	0.587460390
0.010	0.096203758	0.258959190	0.517359390	0.643896400
0.025	0.161842180	0.303526130	0.566394710	0.656799840
0.050	0.269525710	0.375874070	0.650112220	0.749727080

Table 5
Relative errors for scheme (11) at selected mesh points with several grid sizes

h	*(x, t)*			
	(0.25, 0.25)	(0.50, 0.50)	(0.75, 0.75)	(1.00, 1.00)
0.005	0.073964097	0.243110790	0.499832550	0.587404540
0.010	0.095810774	0.257318630	0.514479480	0.604163810
0.025	0.160790680	0.299230670	0.558620850	0.655342320
0.050	0.267276840	0.366743090	0.632617820	0.743583750

Table 6
Comparison of average relative errors between CTCS scheme and scheme (11)

Scheme	*h*			
	0.005	0.010	0.025	0.050
CTCS scheme	0.32842459	0.35450585	0.41870782	0.50469965
Scheme (11)	0.32749333	0.35259906	0.41360435	0.49335045

Table 4, Table 5 and Table 6 demonstrates the relative errors and the average relative errors for the CTCS scheme and the approximate scheme for Example 2 at selected grid sizes respectively. The relative errors of the approximate solution approach zero as grid size reduces. On the other hand, the average relative errors in Table 6 become smaller when grid size approaches zero. These evidences indicate both schemes are convergent.

In addition, as the grid sizes become smaller, the numerical approximate solutions converge to the exact solution. Hence, both schemes are stable and consistent as grid sizes shrink to zero. However, this shows that scheme (11) is more accurate than the CTCS scheme.

4.3. Example 3

As a final problem, we considered the nonlinear inhomogeneous Klein-Gordon equation in [5,12-14,20,32,33]

$$u_{tt} - u_{xx} + u^2 = -x^2 \cos^2(t) \quad , \quad 0 < x < 1 \quad , \quad 0 < t < 1 \tag{12}$$

with initial conditions

$$u(x,0) = 0 \quad , \quad u_t(x,0) = 0 \quad t > 0 \tag{13}$$

The analytical solution of Example 3 is $u(x,t) = \cos(t)$ that can be found in [32].
Here, by using scheme (5), we obtain the new approximate scheme according to Example 3 as follow:

$$
\begin{aligned}
&U_{i+2,j+1} = \\
&U_{i+1,j+2} + U_{i+1,j} - U_{i,j+1} + \left(1-e^{-h}\right)^2 \left(U_{i+1,j+1}\right)^2 - \left(1-e^{-h}\right)^2 \{[4 \cdot \left(-x_{i+1}\cos\left(t_{j+1}\right) + \left(x_{i+1}\right)^2 \cos\left(t_{j+1}\right)^2\right) \\
&\cdot \left(-x_i \cos\left(t_j\right) + \left(x_i\right)^2 \cos\left(t_j\right)^2\right) \cdot \left(-x_{i+1}\cos\left(t_j\right) + \left(x_{i+1}\right)^2 \cos\left(t_j\right)^2\right) \cdot \left(-x_i \cos\left(t_{j+1}\right) + \left(x_i\right)^2 \cos\left(t_{j+1}\right)^2\right)] / \\
&[\left(-x_{i+1}\cos\left(t_{j+1}\right) + \left(x_{i+1}\right)^2 \cos\left(t_{j+1}\right)^2\right) \cdot \left(-x_i \cos\left(t_j\right) + \left(x_i\right)^2 \cos\left(t_j\right)^2\right) \left(\left(-x_{i+1}\cos\left(t_j\right) + \left(x_{i+1}\right)^2 \cos\left(t_j\right)^2\right) + \right. \\
&\left. \left(-x_i \cos\left(t_{j+1}\right) + \left(x_i\right)^2 \cos\left(t_{j+1}\right)^2\right)\right) + \left(-x_{i+1}\cos\left(t_j\right) + \left(x_{i+1}\right)^2 \cos\left(t_j\right)^2\right) \cdot \left(-x_i \cos\left(t_{j+1}\right) + \left(x_i\right)^2 \cos\left(t_{j+1}\right)^2\right) \\
&\left(\left(-x_{i+1}\cos\left(t_{j+1}\right) + \left(x_{i+1}\right)^2 \cos\left(t_{j+1}\right)^2\right) + \left(-x_i \cos\left(t_j\right) + \left(x_i\right)^2 \cos\left(t_j\right)^2\right)\right)]\}
\end{aligned}
\tag{14}
$$

We constructed computer programs for the application of CTCS scheme and scheme (14) for Example 3. The presented numerical results and graphs in Fig. 5 and Fig. 6 illustrate the respective approximate solution and relative errors at selected mesh points with several grid sizes.

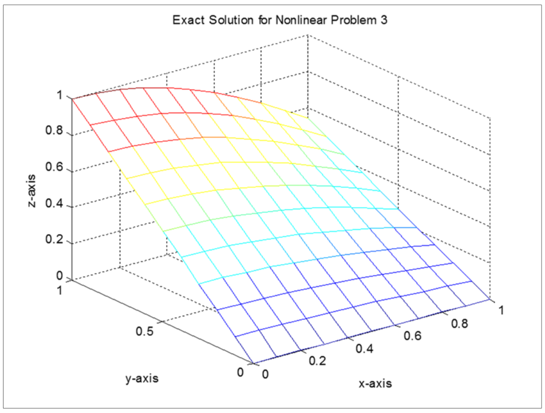

Fig. 5. The Exact solution of Example 3 in graphical form at *h* = 0.05

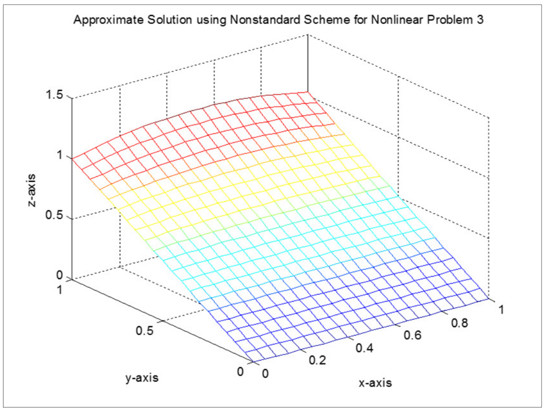

Fig. 6. The Solution of Example 3 in graphical form using scheme (14) at *h* = 0.05

The above graphical illustrations show that the graph for approximate solution in Fig. 6 looks the same as the graph for the exact solution in Fig. 5 due to the occurrence of smaller errors.

Table 7
Relative errors for CTCS scheme at selected mesh points with several grid sizes

h	***(x, t)***			
	(0.25, 0.25)	(0.50, 0.50)	(0.75, 0.75)	(1.00, 1.00)
0.005	0.042563088	0.183836890	0.456806460	0.856050190
0.010	0.042550590	0.183823640	0.457773060	0.861226710
0.025	0.042463092	0.183730870	0.460489320	0.876404740
0.050	0.042150526	0.183399570	0.464408480	0.900508280

Table 8
Relative errors for scheme (14) at selected mesh points with several grid sizes

h	***(x, t)***			
	(0.25, 0.25)	(0.50, 0.50)	(0.75, 0.75)	(1.00, 1.00)
0.005	0.042723226	0.183950610	0.456580210	0.856027990
0.010	0.042837179	0.184010160	0.457295500	0.861137500
0.025	0.042998486	0.183967560	0.459142370	0.875845090
0.050	0.042825644	0.183345530	0.461316130	0.898288220

Table 9
Comparison of average relative errors between CTCS scheme and scheme (14)

Scheme	***h***			
	0.005	0.010	0.025	0.050
CTCS scheme	0.26878149	0.27041902	0.27527585	0.28316757
Scheme (14)	0.26872958	0.27029209	0.27481876	0.28188102

Table 7, Table 8 and Table 9 demonstrates the relative errors and the average relative errors for the CTCS scheme and the approximate scheme for Example 3 at selected grid sizes respectively. The relative errors of the approximate solution approach zero as grid size reduces. Meanwhile, the average relative errors in Table 9 are smaller when grid size reduces to zero. These evidences indicate both schemes are converging.

Furthermore, as the grid sizes become smaller, the numerical approximate solutions closer to the exact solution. Thus, both schemes are stable and consistent as grid size approaches zero. This shows that the scheme (14) is more accurate than the CTCS scheme.

4. Conclusion

The numerical experiments for modified scheme have demonstrated the good performance for selected nonlinear inhomogeneous Klein-Gordon problems. A comparative study on the average relative errors between CTCS scheme and the nonstandard finite difference procedure at selected grid sizes, *h* was done. As the grid sizes become finer, the level of accuracy increases. Hence, we can conclude that the nonstandard finite difference scheme that is associated with harmonic mean averaging in (5) is effective and shows significant improvement in solving the nonlinear Klein-Gordon equations over existing methods. The scheme is also observed to be locally stable and convergent.

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