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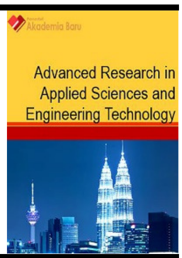

Theoretical analysis of numerical integration rule for nonlinear goursat problem

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R.F. Deraman [1,*], M. Mahayadin [1], S.Z. Mohd Ruslan [1], N.I. Othman [2], M. A.S. Nasir [2]

[1] Faculty of Computer and Mathematical Sciences, Universiti Teknologi MARA Negeri Sembilan Campus, Kuala Pilah Branch,72000 Kuala Pilah, Malaysia
[2] Faculty of Computer and Mathematical Sciences, Universiti Teknologi MARA Selangor Campus, 40450 Shah Alam, Malaysia

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ABSTRACT

Many nonlinear problems that arise in various science and engineering fields can be modelled by the Goursat partial differential equations. Modelling these non-linear problems using the Goursat partial differential equations has not received much attention especially the theoretical aspect . The proposed scheme of solution is supported by examining a nonlinear Goursat problem. The verification of the theoretical results from several series of numerical experiments are discussed. Results obtained from Taylor series expansion show that the proposed new scheme is consistent. By using the von Neumann analysis and essence of stability, the proposed new scheme is found to be unconditionally stable. In addition, the trend of the numerical results shows that the new scheme is also convergent.

1. Introduction

The Goursat partial differential equation is a hyperbolic partial differential equation which is widely applied in the science and engineering fields. Many approaches have been suggested to approximate the solutions of the Goursat partial differential equation such as the finite difference [1], Sumudu Decomposition [2], differential transform [3], variational iteration [4] and Adomian's decomposition [5].

The Goursat partial differential equation has been applied in problems such as micro differential operator [6], isotropic plate [7], van der Waals gas expansion [8], heat conductivity [9], trajectory generation for the N-Trailer [10], supersonic flows [11] and Einstein-Vlasov system [12].

* *Corresponding author.*
E-mail address: rosfadilah7706@ns.uitm.edu.my (R.F. Deraman)

In this paper, we apply the linear operators expansion associated with Newton-Cotes integration rule to solve nonlinear Goursat problem. The theoretical analyses of the linear schemes have been studied in [13] and [14]. The main objective of this paper is to present the theoretical aspects of the proposed scheme for the application of the Goursat partial differential equation on a standard nonlinear problem. Instead of the accuracy aspect, we obtained results relating to the stability, consistency and convergence of the proposed scheme.

2. Methodology

2.1 The goursat problem

The general form of Goursat problem is given in [1] as

$$\begin{aligned} &u_{xy} = f(x, y, u, u_x, u_y), \\ &u(x,0) = g_1(x), \quad u(0, y) = g_2(y), \quad g_1(0) = g_2(0) = u(0,0), \\ &0 \le x \le c, \quad 0 \le y \le d. \end{aligned} \tag{1}$$

The established finite difference scheme with mean averaging of functional values are given by [1] as

$$\frac{u_{i+1,j+1} + u_{i,j} - u_{i+1,j} - u_{i,j+1}}{h^2} = \frac{1}{4}(f_{i+1,j+1} + f_{i,j} + f_{i+1,j} + f_{i,j+1}) \tag{2}$$

where *h* denotes the grid size. Following [15], (1) can be expanded as:

$$u(x, y) = u(x,0) + u(0, y) - u(0,0) + L_x^{-1} L_y^{-1} f(x, y, u, u_x, u_y) \tag{3}$$

Then, the approximation of (3) can be rewritten as below. In the next section, the approximation of the double integral in (4) will be presented.

$$u(x_0 + h, y_0 + h) = u(x_0 + h, y_0) + u(x_0, y_0 + h) - u(x_0, y_0) + \int_{x_0}^{x_0+h} \int_{y_0}^{y_0+h} f(x, y, u, u_x, u_y) dy dx \tag{4}$$

2.2 The linear operators expansion associated with newton-cotes rule for the goursat problem

By utilizing the Newton-Cotes of order two rule in (4), we obtained (5) as presented in [16].

$$\begin{aligned} u_{i+2,j+2} = {} & u_{i+2,j} + u_{i,j+2} - u_{i,j} \\ & + \frac{hk}{9}\left(f_{i,j} + 4 f_{i+1,j} + f_{i+2,j} + f_{i,j+2} + 4 f_{i+1,j+2} + f_{i+2,j+2} \right) \\ & + \frac{4hk}{9}\left(f_{i,j+1} + 4 f_{i+1,j+1} + f_{i+2,j+1} \right), \end{aligned} \tag{5}$$

where *h=k*. Consider the following homogeneous nonlinear Goursat problem given in [1]:

$$
\begin{aligned}
&u_{xy} = e^{2u},\\
&u(x,0) = \frac{x}{2} - \ln\left(1+e^{x}\right),\\
&u(0,y) = \frac{y}{2} - \ln\left(1+e^{y}\right),\\
&0 \le x \le 3.2, \quad 0 \le y \le 3.2.
\end{aligned} \tag{6}
$$

The analytical solution of (6) is $u(x,y) = \frac{x+y}{2} - \text{In}(e^{x}+e^{y})$.

3. Results and discussions

3.1 Consistency

A finite difference scheme is said to be consistent if in the limit, the grid spacings are reduced. However, in the arbitrary manner that allows them to have different orders of magnitude, the finite difference equation approximates the partial differential equation with increasing accuracy. The consistency can be achieved if all variables are continuous functions. The finite difference scheme is consistent if the modified differential equation reduces to the original partial differential equation as step size tends to zero. In demonstrating the consistency, scheme (5) can be rewritten as

$$
\begin{aligned}
&u_{i+2,j+2} - u_{i+2,j} - u_{i,j+2} + u_{i,j}\\
&\quad = \frac{4h^2}{9}\left(\frac{1}{4}e^{2u_{i,j}} + e^{2u_{i+1,j}} + \frac{1}{4}e^{2u_{i+2,j}} + \frac{1}{4}e^{2u_{i,j+2}} + e^{2u_{i+1,j+2}}\right.\\
&\quad \left. + \frac{1}{4}e^{2u_{i+2,j+2}} + e^{2u_{i,j+1}} + 4e^{2u_{i+1,j+1}} + e^{2u_{i+2,j+1}}\right).
\end{aligned} \tag{7}
$$

Substituting the exact solution into (7) yields

$$
\begin{aligned}
&u\left(x_{i+2}, y_{j+2}\right) - u\left(x_{i+2}, y_{j}\right) - u\left(x_{i}, y_{j+2}\right) + u\left(x_{i}, y_{j}\right)\\
&\quad = \frac{4h^2}{9}\left[\frac{1}{4}e^{2u\left(x_i, y_j\right)} + e^{2u\left(x_{i+1}, y_j\right)} + \frac{1}{4}e^{2u\left(x_{i+2}, y_j\right)}\right.\\
&\quad + \frac{1}{4}e^{2u\left(x_i, y_{j+2}\right)} + e^{2u\left(x_{i+1}, y_{j+2}\right)} + \frac{1}{4}e^{2u\left(x_{i+2}, y_{j+2}\right)}\\
&\quad \left. + e^{2u\left(x_i, y_{j+1}\right)} + 4e^{2u\left(x_{i+1}, y_{j+1}\right)} + e^{2u\left(x_{i+2}, y_{j+1}\right)}\right].
\end{aligned} \tag{8}
$$

Expanding (8) as a Taylor series about the point (x_i, y_j) the modified differential equation

$$
\begin{aligned}
&\left\{\left[u + 2hu_x + 2hu_y + 2\left(h^2u_{xx} + 2h^2u_{xy} + h^2u_{yy}\right) + \ldots\right]\right.\\
&\left. -\left(u + 2hu_x + 2h^2u_{xx} + \ldots\right) - \left(u + 2hu_y + 2h^2u_{yy} + \ldots\right) + u\right\}
\end{aligned}
$$

$$= \frac{4h^2}{9}\left\{ \frac{1}{4}e^{2u} + e^{2\left(u+hu_x+\frac{h^2}{2}u_{xx}+\ldots\right)} + \frac{1}{4}e^{2\left(u+2hu_x+2h^2u_{xx}+\ldots\right)} \right.$$

$$+\frac{1}{4}e^{2\left(u+2hu_y+2h^2u_{yy}+\ldots\right)} + e^{2\left(u+hu_x+2hu_y+\frac{h^2}{2}u_{xx}+2h^2u_{yy}+\ldots\right)}$$

$$+\frac{1}{4}e^{2\left[u+2hu_x+2hu_y+2\left(h^2u_{xx}+2h^2u_{xy}+h^2u_{yy}\right)+\ldots\right]} + e^{2\left(u+hu_y+\frac{h^2}{2}u_{yy}+\ldots\right)}$$

$$\left. +4e^{2\left[u+hu_x+hu_y+\frac{1}{2}\left(h^2u_{xx}+2h^2u_{xy}+h^2u_{yy}\right)+\ldots\right]} + e^{2\left(u+2hu_x+hu_y+2h^2u_{xx}+\frac{h^2}{2}u_{yy}+\ldots\right)} \right\}, \tag{9}$$

where all terms involving *u* are evaluated at (x_i, y_j). As $h \to 0$, then (9) can be simplified as $u_{xy} = e^{2u}$.

3.2 Stability

It is known that the stability analysis of the nonlinear problem is very complicated and there are no general methods or evidences to analyze the nonlinear case [17] . Based on the detailed investigation, we found that this problem can be von Neumann analysis and convergence analysis (essence of stability). The von Neumann method is most commonly used for the linear case only. However, [18] stated that the method can also be applied for the nonlinear case which is temporarily frozen. Thus, the stability analysis of the proposed scheme (5) for the nonlinear problem (6) after the freezing process is

$$u_{i+2,j+2} = u_{i+2,j} + u_{i,j+2} - u_{i,j}. \tag{10}$$

The error is given by

$$\xi_{i+2,j+2} = \xi_{i+2,j} + \xi_{i,j+2} - \xi_{i,j}. \tag{11}$$

Substituting the error equation $\xi_{i,j} = \lambda^j e^{\sqrt{-1}\,\theta\, i}$ into (10) gives

$$\lambda^{j+2}e^{\sqrt{-1}\theta(i+2)} = \lambda^j e^{\sqrt{-1}\theta(i+2)} + \lambda^{j+2}e^{\sqrt{-1}\theta i} - \lambda^j e^{\sqrt{-1}\theta i}. \tag{12}$$

Due to the difficulty in manipulating the equations the Maple 12 has been used to execute the algebraic manipulation in order to obtain (12) and the stability region for scheme (5), which is shown in Fig. 1.

The shaded region in Fig. 1 shows that scheme (5) is unconditionally stable in approximating nonlinear problem (6). In addition, the stability for finite difference schemes involving homogeneous initial value problem studied in [19] noted that if a scheme is convergent, as the approximation scheme converges to exact solution, then certainly it is bounded in some sense.

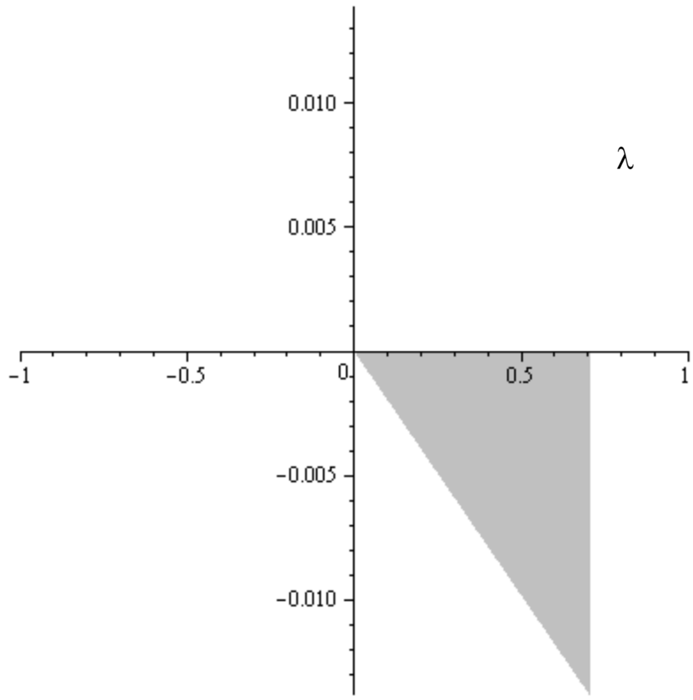

Fig. 1. Stability Region for Scheme (5)

3.4 Convergence

However, there is no specific method to analyse the convergence of nonlinear problems. Therefore, the convergence is determined by observing the trend of the numerical results that have been obtained, as in this case, by observing the trend of the graph of average relative error based on a variety of step sizes as illustrated in Fig. 2.

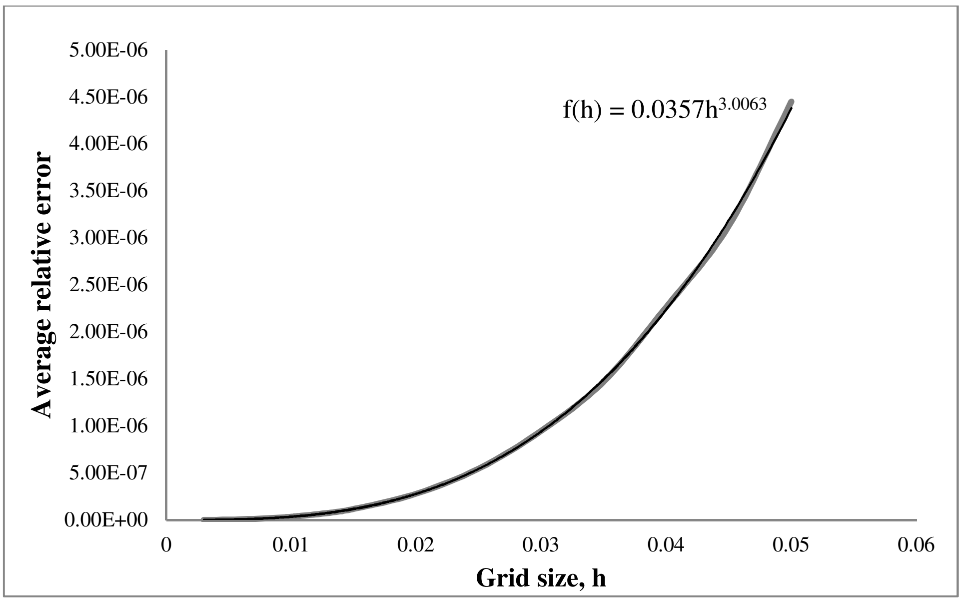

Fig. 2. Convergence Analysis for Scheme (5)

It can be clearly seen that scheme (5) approaches the exact solution of problem (6) as grid spacing tends to zero or when the error becomes smaller as *h* approaches zero. Therefore, we can conclude the proposed scheme (5) is convergent. when compared to Hamilton and Crosser's thermal conductivity model.

4. Conclusion

In conclusion, the theoretical aspects of the finite difference solution of a linear Goursat problem using linear operators expansion associated with Newton-Cotes rule of order two have successfully been presented. The analyses of the numerical results show that the scheme is unconditionally stable, consistent and convergent.

References

[1] Wazwaz A.M., Partial Differential Equations, CRC Press, 2002.

[2] Ahmad, Jamshad, S. Rubab, M. Naeem, and L. A. Tahir. "Application of Sumudu decomposition method to Goursat problem." *Journal of Science and Arts* 14, no. 4 (2014): 287.

[3] Taghvafard, H., and H. G. Erjaee. "Two-Dimensional Differential Transform Method for Solving Linear and Non-Linear Goursat problem." *International Journal for Engineering and Mathematical Sciences* 6, no. 2 (2010): 103-106.

[4] Wazwaz, Abdul-Majid. "The variational iteration method for a reliable treatment of the linear and the nonlinear Goursat problem." *Applied Mathematics and Computation* 193, no. 2 (2007): 455-462.

[5] Ahmad, Jamshad, and Mariyam Mushtaq. "Exact Solution of Linear and Non-linear Goursat Problems." *Universal Journal of Computational Mathematics* 3, no. 1 (2015): 14-17.

[6] Yamazaki, Susumu. "Goursat problem for a microdifferential operator of Fuchsian type and its application." *Publications of the Research Institute for Mathematical Sciences* 33, no. 4 (1997): 599-641.

[7] Aseeri, S. A. "Goursat functions for a problem of an isotropic plate with a curvilinear hole." *Int. J. Open Problems Compt. Math* 1, no. 3 (2008).

[8] Lai, Geng. "On the expansion of a wedge of van der Waals gas into a vacuum." *Journal of Differential Equations* 259, no. 3 (2015): 1181-1202.

[9] oglu Mamedov, Ilgar Gurbat. "3D Goursat problem in the non-classical treatment for Manjeron generalized equation with non-smooth coefficients." *Applied and Computational Mathematics* 4, no. 1-1 (2015): 1-5.

[10] Tilbury, Dawn, Richard M. Murray, and S. Shankar Sastry. "Trajectory generation for the N-trailer problem using Goursat normal form." *IEEE Transactions on Automatic Control* 40, no. 5 (1995): 802-819.

[11] Chen, S. X., and D. N. Li. "Supersonic flow past a symmetrically curved cone." *Indiana University Mathematics Journal* 49, no. 4 (2000): 1411-1435.

[12] Calvin, Tadmon. "The Goursat Problem for the Einstein-Vlasov System:(I) The Initial Data Constraints." *arXiv preprint arXiv:1109.6844* (2011).

[13] Ismail, Md, Ahmad Izani, Salim Nasir, and Mohd Agos. "Numerical solution of a linear Goursat problem stability, consistency and convergence." *WSEAS Transactions on Mathematics* 3, no. 3 (2004).

[14] Nasir, Mohd Agos Salim, and Ahmad Izani Md Ismail. "A fourth-order compact finite difference scheme for the Goursat problem." *Sains Malaysiana* 42, no. 3 (2013): 341-346.

[15] Wazwaz, Abdul-Majid. "The decomposition method for approximate solution of the Goursat problem." *Applied Mathematics and Computation* 69, no. 2 (1995): 299-311.

[16] Deraman, Ros Fadilah, Mohd Agos Salim Nasir, Seripah Awang Kechil, and Ahmad Sukri Aziz. "Adomian decomposition associated with Newton-Cotes formula for solving Goursat problem." In *Proceedings of the 20th National Symposium on Mathematical Sciences: Research in Mathematical Sciences: A Catalyst for Creativity and Innovation*, vol. 1522, no. 1, pp. 476-482. AIP Publishing, 2013.

[17] Yang, Xiao-Zhong, and Na Guo. "A new kind of universal difference scheme for solving nonlinear Sine-Gordon equation."

[18] Fletcher C., Computational Techniques for Fluid Dynamics, Vol. 1, Springer-Verlag Berlin Heidelberg, 1990.

[19] Strikwerda, John C. Finite difference schemes and partial differential equations. Siam, 2004.