Penerbit Akademia Baru

Journal of Advanced Research in Applied Mechanics
ISSN (online): 2289-7895 | Vol. 5, No.1. Pages 15-29, 2015

Constitutive Model for Anterior Leaflet of Heart with Physical Invariants

B. A. Mahad [*,1,a], *M. H. B. M. Shariff* [3,b], *M. Isa* [1,c] *and Z. A. Aziz* [1,2,d]

[1]Department of Mathematical Sciences, Faculty of Science,
[2]UTM Centre for Industrial and Applied Mathematics.
Universiti Teknologi Malaysia, 81310
Skudai, Johor Bahru, Malaysia
[3]Khalifa University of Science, Technology and Research
PO Box 573, Sharjah, United Arab Amirates
[a,*]*mahad@utm.my*, [b]*shariff@kustar.ac.ae*, [c]*mukheta@utm.my*, [d]*zainalaz@utm.my*

Abstract – *Much research has been done in determining constitutive models for Nonlinear Transversely Isotropic Solids. The strain energy functions with different types of invariants were developed in the past to serve some purposes. In isotropic elasticity, phenomenological strain energy functions with principal stretches have certain attractive features from both the mathematical and physical viewpoints. These forms of strain energy have been widely and successfully used in predicting elastic deformations. In this paper, we extend these successful principal-stretches-isotropic models to characterise transversely isotropic solids based on previous work. We introduce five invariants that have immediate physical interpretation. Three of the invariants are the principal extension ratios and the other two are the cosines of the angles between the principal directions of the right stretch tensor and the material preferred direction. This model has an experimental advantage and the theory is compared well with experimental data.*

Keywords: Constitutive model, Transversely Isotropic, Nonlinear Elasticity, Principal Stretches

1.0 INTRODUCTION

Hyperelasticity is the capability of a material to experience large elastic strain due to small forces, without losing its original properties [1]. A hyperelastic material has a nonlinear behaviour, which means that its answer to the load is not directly proportional to the deformation.

The modeling and design of hyperelastic materials consists of the selection of an appropriate strain energy function W and accurate determination of isotropic, hyperelastic materials, therefore extending these models to include anisotropic, pseudoelastic behaviour creates models appropriate for biological tissues. The essential concept of this class of theory is that the energy density in the material can be determined as a function of the strain state. Once the strain-energy function W is known, the stress state can be determined by taking the derivative of W with respect to a strain measure, such as

Journal of Advanced Research in Applied Mechanics
ISSN (online): 2289-7895 | Vol. 5, No.1. Pages 15-29, 2015

$$\sigma = \frac{\partial W}{\partial \varepsilon} \tag{1}$$

where σ is the Cauchy (true) stress tensor and ε is the Green strain tensor. The most common form used to determine stresses for the materials considered here is

$$\sigma = -pI + 2F \frac{\partial W}{\partial C} F^T \tag{2}$$

where F is the deformation gradient, C is the left Cauchy-Green strain tensor, p is the Lagrange multiplier to enforce incompressibility, and I is the identity tensor [2]. The basic physical properties of biological tissue govern the assumptions that can be made in the formulation of a constitutive model. Many models exist that involve assuming a strain-energy function for the tissue. Based on observations for rat mesentery, Fung and co-workers [3] proposed that the strain-energy should be exponentially related to the strain. In transverse isotropy, the material has one preferred direction parallel to the fiber direction, and the responses in every direction perpendicular to the referred direction are identical to each other.

1.1 Transversely Isotropic Model

In general (in three dimensions), two independent invariants are sufficient to characterize the anisotropic nature of a transversely isotropic material model, one of which is related directly to the fiber stretch and is denoted by I_4. The standard reinforcing model is a quadratic function that depends only on this invariant. The other invariant, denoted by I_5, is also related to the fiber stretch but introduces an additional effect that relates to the behaviour of the reinforcement under shear deformations. When the deformation is restricted to plane strain with the fiber direction in the considered plane these two invariants are no longer independent [4,5]. The models in this section determine strain-energy functions based on the assumption of transverse isotropy and in terms of strain invariants. Transverse hyperelasticity can be completely described by the three strain invariants and two pseudo-invariants [2].

In [6], for transversely isotropic the fourth strain invariants is included in energy function, i.e. $W(I_1, I_2, I_3, I_4)$. λ_i is denoted as principal stretches of deformation gradient F and I_i is a function of a strain-energy function for transversely isotropic solid i.e. mitral valve tissue was carefully determined and verified by [7,8]. In the past, the stress deformation response was shown to be chiefly a function of the first invariant and the stretch in the fiber direction,

$$W = W(I_1, \alpha) \tag{3}$$

Specifically, the response was modeled by a form analogous to the exponential proposed by [9],

$$W(I_1, I_4) = c_0 \left\{ \exp\left[c_1 (I_1 - 3)^2 + c_2 \left(I_4^{1/2} - I \right)^4 \right] - 1 \right\} \tag{4}$$

where c_0, c_1, c_2 are constants fit to the experimental data, and they used pseudo-invariant that defined in terms of the Cauchy-Green strain by [10] such as

$$I_4 = N \cdot C \cdot N = \alpha^2 \tag{5}$$

Journal of Advanced Research in Applied Mechanics
ISSN (online): 2289-7895 | Vol. 5, No.1. Pages 15-29, 2015

to substitute I_4 for α. The interior and posterior leaflets have slightly different responses, reflected by the difference in values for the three constants. The strain-energy function in above along with the coefficient values accurately predicts the stress deformation behaviour of biological tissues i.e. the leaflet tissue and also any other author such that [11-15] but not all their strain-energy function have immediate physical interpretation and the constitutive equation does not experimental friendly.

We proposed to form the strain-energy function of transversely isotropic solids have immediate physical interpretation by introduce five invariants. Three of the invariants are the principal extension ratios $\lambda_i > 0\,(i = 1, 2, 3)$ and the other two are $I \geq \zeta_1 = (a \cdot e_1)^2 \geq 0$ and $I \geq \zeta_2 = (a \cdot e_2)^2 \geq 0$, where e_1 and e_2 are any two of the principal directions of the right stretch tensor U. The physical meaning of λ_i is obvious and it is clear that $a \cdot e_i\ (i = 1, 2)$ is the cosine of the angle between the principal direction e_i and the preferred direction a.

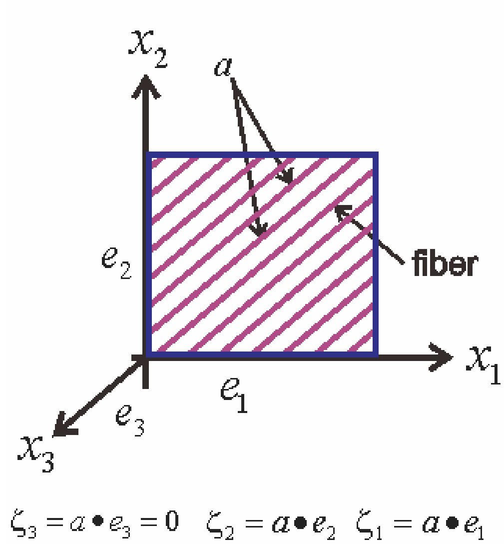

Figure 1: Modelling for cosine of angle between the principal direction e_i and the preferred direction a

It is hoped that strain-energy functions of transversely isotropic elastic solids which depend explicitly on the variables $\lambda_1, \lambda_2, \lambda_3, \zeta_1$ and ζ_2 may achieve the same success as strain-energy functions of isotropic elastic solids which depend explicitly on λ_1, λ_2 and λ_3. To obtain a specific form of the strain-energy from an experiment, it is convenient to have explicit and analytic expressions for the five derivatives of the strain-energy function with respect to its invariants. We will also show that a strain-energy function written in terms of the proposed variables enjoys a symmetry and orthogonal properties similar to the symmetry possessed by a strain-energy function of an isotropic elastic solid written in terms of principal stretches.

1.2 Continuum Mechanics

Journal of Advanced Research in Applied Mechanics
ISSN (online): 2289-7895 | Vol. 5, No.1. Pages 15-29, 2015

We consider an elastic material for which the material properties are characterized in terms of a strain-energy (per unit volume), denoted $W = W(F)$ and defined on the space of deformation gradients. This theory is known as hyperelasticity. For an inhomogeneous material, i.e. One whose properties vary from point to point, W depends on X in addition to F, but we do not indicate this dependence explicitly in what follows.

For an unconstrained hyperelastic material the nominal stress is given by

$$S = H(F) = \frac{\partial W}{\partial F}, \tag{6}$$

where the notation H is defined. The tensor function H is referred to as the response function of the material relative to the deformed configuration, B_r, in respect of the nominal stress tensor. In components, the derivative in (6) is written $S_{\alpha i} = \frac{\partial W}{\partial F_{i\alpha}}$, which provides our convention for ordering of the indices in the partial derivative with respect to F.
For an incompressible material the counterpart of (6) is

$$S = \frac{\partial W}{\partial F} - pF^{-1}, \qquad \det F = 1 \tag{7}$$

where p is the Lagrange multiplier associated with the incompressibility constraint and is referred to as the arbitrary hydrostatic pressure.

The Cauchy stress tensor corresponding to (6), then to be given by

$$\sigma = G(F) \equiv J^{-1} F \frac{\partial W}{\partial F} \tag{8}$$

Wherein the response function G associated with σ is defined. As for H, the form of G depends on the choice of reference configuration, and G is referred to as the response function of the material relative to B_r associated with the Cauchy stress tensor. Unlike H, however, G is a symmetric tensor-valued function. For incompressible materials (8) is replaced by

$$\sigma = F \frac{\partial W}{\partial F} - pI, \qquad \det F = 1 \tag{9}$$

2.0 Strain Energy Function with Physical Invariants

Let $F = \frac{\partial x}{\partial X}$ be the deformation gradient tensor, where X is the position vector of a material particle in the undeformed configuration and x is the corresponding position vector in the deformed configuration. The right Cauchy-Green deformation tensor, denoted C, is given by $C = F^T F$ and I_1, I_2 and I_3 are its principal invariants, which are given by

$$I_1 = \operatorname{tr} C, \qquad I_2 = \frac{1}{2}\left[(\operatorname{tr} C)^2 - \operatorname{tr}\left(C^2\right)\right], \qquad I_3 = \det C = (\det F)^2 \tag{10}$$

Journal of Advanced Research in Applied Mechanics
ISSN (online): 2289-7895 | Vol. 5, No.1. Pages 15-29, 2015

Let the unit vector A define the direction of the fiber reinforcement in the undeformed configuration. Then, additional invariants, denoted I_4 and I_5, that couple A and C are given by

$$I_4 = (FA)\cdot(FA) = A\cdot(CA), \qquad I_5 = A\cdot\left(C^2 A\right) \tag{11}$$

We extend the above principal stretches isotropic models to characterize transversely isotropic solids where the principal stretch λ_i $(i = 1, 2, 3)$ is given by

$$\lambda_i = \sqrt{e_i \cdot U^2 e_i} \tag{12}$$

where $U^2 = F^T F$ and e_i is a principal direction of U. In this paper, all subscripts i and j take the values 1, 2 and 3, unless stated otherwise.

The material response of a transversely isotropic solid is indifferent to arbitrary rotations about the direction a and by replacement of a by $-a$. Following [16], such materials can be characterized with a strain-energy function W_e which depends on U and the tensor $A = a \otimes a$ ($\otimes$ denotes the dyadic product), i.e.,

$$W_e = \hat{W}(U, A) \tag{13}$$

Since

$$U = \lambda_1 E_1 + \lambda_2 E_2 + \lambda_3 E_3 \tag{14}$$

where $E_i = e_i \otimes e_i$. We can express

$$\hat{W}(U, A) = \tilde{W}(\lambda_1, \lambda_2, \lambda_3, E_1, E_2, E_3, A). \tag{15}$$

[2] has shown that the strain-energy function can be written in the form

$$W_e = W_f(\lambda_1, \lambda_2, \lambda_3, \zeta_1, \zeta_2, \zeta_3). \tag{16}$$

The function W_f enjoys the symmetrical property [2]

$$W_f(\lambda_1, \lambda_2, \lambda_3, \zeta_1, \zeta_2, \zeta_3) = W_f(\lambda_2, \lambda_1, \lambda_3, \zeta_2, \zeta_1, \zeta_3) = W_f(\lambda_3, \lambda_2, \lambda_1, \zeta_3, \zeta_2, \zeta_1) \tag{17}$$

However, ζ_3 depends on ζ_1 and ζ_2, i.e.,

$$\zeta_3 = 1 - \zeta_1 - \zeta_2 \tag{18}$$

Hence, we can omit ζ_3 from the list in Equation (16) and we then have

$$W_e = \tilde{W}(\lambda_1, \lambda_2, \lambda_3, \zeta_1, \zeta_2) = W_f(\lambda_1, \lambda_2, \lambda_3, \zeta_1, \zeta_2, 1 - \zeta_1 - \zeta_2) \tag{19}$$

Journal of Advanced Research in Applied Mechanics
ISSN (online): 2289-7895 | Vol. 5, No.1. Pages 15-29, 2015

The commonly used invariants can be written explicitly in terms of the physical variables, i.e.,

$$I_1=\lambda_1^2+\lambda_2^2+\lambda_3^2,\quad I_2=\lambda_1^2\lambda_2^2+\lambda_1^2\lambda_3^2+\lambda_2^2\lambda_3^2,\quad I_3=(\lambda_1\lambda_2\lambda_3)^2$$
$$I_4=\lambda_1^2\zeta_1+\lambda_2^2\zeta_2+\lambda_3^2\zeta_3,\quad I_5=\lambda_1^4\zeta_1+\lambda_2^4\zeta_2+\lambda_3^4\zeta_3 \tag{20}$$

For an incompressible material $\lambda_1\lambda_2\lambda_3=1$, the number of variables is reduce to 4 and we can express

$$W_e=W(\lambda_1,\lambda_2,\zeta_1,\zeta_2)=\tilde{W}\left(\lambda_1,\lambda_2,\frac{1}{\lambda_1\lambda_2},\zeta_1,\zeta_2\right) \tag{21}$$

In the reference state $U=I,\lambda_1=\lambda_2=\lambda_3=1$, any orthonormal set of vectors can represent the principal directions of U. For simplicity, we let $a=e_3$ and it is clear that $\zeta_3=1,\zeta_1=\zeta_2=0$ in this state. To be consistent with the classical linear theory of incompressible transversely isotropic elasticity, appropriate for infinitesimal deformations, we must have the relations

$$\frac{\partial^2 W}{\partial\lambda_1^2}(1,1,0,0)=\frac{\partial^2 W}{\partial\lambda_2^2}(1,1,0,0)=4\mu_L+\beta,$$
$$\frac{\partial^2 W}{\partial\lambda_1\partial\lambda_2}(1,1,0,0)=4\mu_L-2\mu_T+\beta,$$
$$\frac{\partial^2 W}{\partial\lambda_i\partial\zeta_j}(1,1,0,0)=\frac{\partial^2 W}{\partial\zeta_i\partial\zeta_j}(1,1,0,0)=0,\qquad i,\ j=1,2. \tag{22}$$

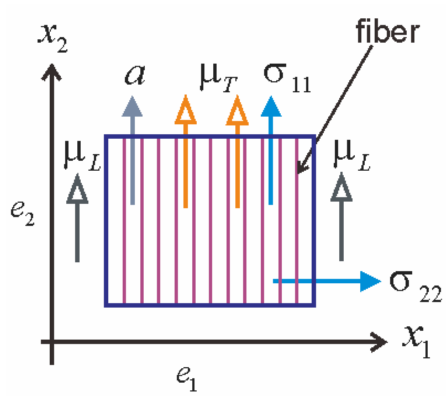

Figure 2: μ_T and μ_L, represent the elastic shear moduli in the ground state and β can be related to other elastic constant which has more direct physical interpretation, such as the extension modulus.

2.1 Stress –Strain for Biological Soft Tissues

Using series expansion techniques, the strain-energy function can be written as

$$W_e=\sum_{i=1}^{3}\hat{f}(\lambda_i,\zeta_i)+\hat{g}(\lambda_1,\lambda_2,\zeta_1,\zeta_2)+\hat{g}(\lambda_1,\lambda_3,\zeta_1,\zeta_3)+\hat{g}(\lambda_2,\lambda_3,\zeta_2,\zeta_3) \tag{23}$$

Journal of Advanced Research in Applied Mechanics
ISSN (online): 2289-7895 | Vol. 5, No.1. Pages 15-29, 2015

where $\lambda_3 = \dfrac{1}{\lambda_1 \lambda_2}$, the function $\hat{g}$ has the symmetry $\hat{g}(x, y, \phi, \varphi) = \hat{g}(y, x, \varphi, \phi)$. A special case of (21) is augmented form

$$W_e = W_{iso}(\lambda_1, \lambda_2, \lambda_3) + W_{trn}(\lambda_1, \lambda_2, \lambda_3, \zeta_1, \zeta_2, \zeta_3) \quad (24)$$

Where

$$W_{iso}(\lambda_1, \lambda_2, \lambda_3) = \sum_{i=1}^{3} r(\lambda_i) + \bar{g}(\lambda_1, \lambda_2) + \bar{g}(\lambda_1, \lambda_3) + \bar{g}(\lambda_2, \lambda_3)$$

$$W_{trn}(\lambda_1, \lambda_2, \lambda_3, \zeta_1, \zeta_2, \zeta_3) = \sum_{i=1}^{3} f(\lambda_i, \zeta_i) + g(\lambda_1, \lambda_2, \zeta_1, \zeta_2) + g(\lambda_1, \lambda_3, \zeta_1, \zeta_3) + g(\lambda_2, \lambda_3, \zeta_2, \zeta_3)$$

g has the same symmetry property as $\hat{g}$ and $\bar{g}(x, y) = \bar{g}(y, x)$. W_{iso} is a strain-energy function for an isotropic material. A special form of the augmented strain-energy done by Shariff [17,18] with its isotropic base taking the Valanis & Landel model to the semi-linear form

$$W_e = \sum_{i=1}^{3} \mu_T (\lambda_i - 1)^2 + 2(\mu_L - \mu_T) \sum_{i=1}^{3} \zeta_i (\lambda_i - 1)^2 + \frac{\beta}{2} \sum_{i,j=1}^{3} \zeta_i \zeta_j (\lambda_i - 1)(\lambda_j - 1) \quad (25)$$

for mathematical simplicity, we proposed the special form of W_e which is linear in its parameters, i.e.,

$$W_e = \sum_{i=1}^{3} \mu_T r(\lambda_i) + 2(\mu_L - \mu_T) \sum_{i=1}^{3} \zeta_i s(\lambda_i) + \frac{\beta}{2} \sum_{i,j=1}^{3} \zeta_i \zeta_j t(\lambda_i)\, t(\lambda_j) \quad (26)$$

For an incompressible material, we have

$$\sigma_{11} = \lambda_1 \frac{\partial W}{\partial \lambda_1}, \quad \sigma_{22} = \lambda_2 \frac{\partial W}{\partial \lambda_2}, \quad \sigma_{12} = 2\left(\frac{\partial W}{\partial \zeta_1} - \frac{\partial W}{\partial \zeta_2}\right) \frac{\lambda_1 \lambda_2 e_1 \cdot A e_2}{\lambda_1^2 - \lambda_2^2},$$
$$\sigma_{13} = 2\left(\frac{\partial W}{\partial \zeta_1} - \frac{\partial W}{\partial \zeta_3}\right) \frac{\lambda_1 \lambda_3 e_1 \cdot A e_3}{\lambda_1^2 - \lambda_3^2}, \quad \sigma_{23} = 2\left(\frac{\partial W}{\partial \zeta_1} - \frac{\partial W}{\partial \zeta_2}\right) \frac{\lambda_1 \lambda_2 e_1 \cdot A e_2}{\lambda_1^2 - \lambda_2^2}, \quad (27)$$

Where $A = a \otimes a$, $\sigma_{12} = \sigma_{13} = \sigma_{23} = 0$ since $e_1 \perp e_2, e_1 \perp e_3, e_2 \perp e_3$. Preferred direction is a perpendicular to e_3, we have $\zeta_3 = 0$ and $\zeta_2 = 1 - \zeta_1$, where $\zeta_3 = 1 - \zeta_1 - \zeta_2$.

The strain-energy function can be expressed as

Journal of Advanced Research in Applied Mechanics
ISSN (online): 2289-7895 | Vol. 5, No.1. Pages 15-29, 2015

$$W(\lambda_1,\lambda_2,\zeta_1,\zeta_2)=f(\lambda_1)\left[\mu_T+2(\mu_L-\mu_T)\zeta_1+\frac{\beta}{2}\zeta_1^2\right]+ f(\lambda_2)\left[\mu_T+2(\mu_L-\mu_T)\zeta_2+\frac{\beta}{2}\zeta_2^2\right]+f(\lambda_3)\left[\mu_T+2(\mu_L-\mu_T)\zeta_3+\frac{\beta}{2}\zeta_3^2\right]+ \frac{\beta}{2}[2\zeta_1\zeta_2 s(\lambda_1)s(\lambda_2)+2\zeta_1\zeta_3 s(\lambda_1)s(\lambda_3)+2\zeta_2\zeta_3 s(\lambda_2)s(\lambda_3)] \tag{28}$$

For incompressible material , $\lambda_1\,\lambda_2\,\lambda_3=1$, then $\lambda_3=\dfrac{1}{\lambda_1\lambda_2}$. After differentiate equation (28) with respect to λ_1, we will obtain:

$$\frac{\partial W}{\partial\lambda_1}=f'(\lambda_1)\left[\mu_T+2(\mu_L-\mu_T)\zeta_1+\frac{\beta}{2}\zeta_1^2\right]+f'(\lambda_3)\left(-\frac{1}{\lambda_1^2\lambda_2}\right)\left[\mu_T+2(\mu_L-\mu_T)\zeta_3+\frac{\beta}{2}\zeta_3^2\right]+ \beta\left[\zeta_1\zeta_2 s'(\lambda_1)s(\lambda_2)+\zeta_1\zeta_3\left(s'(\lambda_1)s(\lambda_3)+s(\lambda_1)s'(\lambda_3)\left(-\frac{1}{\lambda_1^2\lambda_2}\right)\right)+\zeta_2\zeta_3 s(\lambda_2)s'(\lambda_3)\left(-\frac{1}{\lambda_1^2\lambda_2}\right)\right]$$

Then,

$$\sigma_{11}=\lambda_1\frac{\partial W}{\partial\lambda_1}=\lambda_1 f'(\lambda_1)\left[\mu_T+2(\mu_L-\mu_T)\zeta_1+\frac{\beta}{2}\zeta_1^2\right]+\lambda_1 f'(\lambda_3)\left(-\frac{1}{\lambda_1^2\lambda_2}\right)\left[\mu_T+2(\mu_L-\mu_T)\zeta_3+\frac{\beta}{2}\zeta_3^2\right]+ \beta\left[\zeta_1\zeta_2\lambda_1 s'(\lambda_1)s(\lambda_2)+\zeta_1\zeta_3\left(\lambda_1 s'(\lambda_1)s(\lambda_3)+\lambda_1 s(\lambda_1)s'(\lambda_3)\left(-\frac{1}{\lambda_1^2\lambda_2}\right)\right)+\zeta_2\zeta_3\lambda_1 s(\lambda_2)s'(\lambda_3)\left(-\frac{1}{\lambda_1^2\lambda_2}\right)\right] \tag{29}$$

Substitute $\lambda_3=\dfrac{1}{\lambda_1\,\lambda_2}$ into third terms of equation (29) we obtain:

$$\sigma_{11}=\lambda_1\frac{\partial W}{\partial\lambda_1}=\lambda_1 f'(\lambda_1)\left[\mu_T+2(\mu_L-\mu_T)\zeta_1+\frac{\beta}{2}\zeta_1^2\right]- \lambda_3 f'(\lambda_3)\left[\mu_T+2(\mu_L-\mu_T)\zeta_3+\frac{\beta}{2}\zeta_3^2\right]+ \beta[\zeta_1\zeta_2\lambda_1 s'(\lambda_1)s(\lambda_2)+\zeta_1\zeta_3(\lambda_1 s'(\lambda_1)s(\lambda_3)-\lambda_3 s(\lambda_1)s'(\lambda_3))+\zeta_2\zeta_3\lambda_3 s(\lambda_2)s'(\lambda_3)] \tag{30}$$

Similarly integrate equation (28) with respect to λ_2, we obtain:

$$\frac{\partial W}{\partial\lambda_2}=f'(\lambda_2)\left[\mu_T+2(\mu_L-\mu_T)\zeta_2+\frac{\beta}{2}\zeta_2^2\right]+f'(\lambda_3)\left(-\frac{1}{\lambda_1\lambda_2^2}\right)\left[\mu_T+2(\mu_L-\mu_T)\zeta_3+\frac{\beta}{2}\zeta_3^2\right]+ \beta\left[\zeta_1\zeta_2 s(\lambda_1)s'(\lambda_2)+\zeta_1\zeta_3 s(\lambda_1)s'(\lambda_3)\left(-\frac{1}{\lambda_1\lambda_2^2}\right)+\zeta_2\zeta_3\left(s'(\lambda_2)s(\lambda_3)+s(\lambda_2)s'(\lambda_3)\left(-\frac{1}{\lambda_1\lambda_2^2}\right)\right)\right] \tag{31}$$

Journal of Advanced Research in Applied Mechanics
ISSN (online): 2289-7895 I Vol. 5, No.1. Pages 15-29, 2015

$$\sigma_{22} = \lambda_2 \frac{\partial W}{\partial \lambda_2} = \lambda_2 f'(\lambda_2)\left[\mu_T + 2(\mu_L - \mu_T)\zeta_2 + \frac{\beta}{2}\zeta_2^2\right] - \lambda_2 f'(\lambda_3)\left(-\frac{1}{\lambda_1\lambda_2^2}\right)\left[\mu_T + 2(\mu_L - \mu_T)\zeta_3 + \frac{\beta}{2}\zeta_3^2\right] + \beta\left[\begin{array}{l}\zeta_1\zeta_2\lambda_2 s(\lambda_1)s'(\lambda_2) + \zeta_1\zeta_3\lambda_2 s(\lambda_1)s'(\lambda_3)\left(-\frac{1}{\lambda_1\lambda_2^2}\right) + \\ \zeta_2\zeta_3\left(\lambda_2 s'(\lambda_2)s(\lambda_3) + \lambda_2 s(\lambda_2)s'(\lambda_3)\left(-\frac{1}{\lambda_1\lambda_2^2}\right)\right)\end{array}\right] \quad (32)$$

Substitute $\lambda_3 = \frac{1}{\lambda_1 \lambda_2}$ into third terms of equation (32) we obtain:

$$\sigma_{22} = \lambda_2 f'(\lambda_2)\left[\mu_T + 2(\mu_L - \mu_T)\zeta_2 + \frac{\beta}{2}\zeta_2^2\right] - \lambda_3 f'(\lambda_3)\left[\mu_T + 2(\mu_L - \mu_T)\zeta_3 + \frac{\beta}{2}\zeta_3^2\right] + \beta[\zeta_1\zeta_2\lambda_2 s(\lambda_1)s'(\lambda_2) - \zeta_1\zeta_3\lambda_3 s(\lambda_1)s'(\lambda_3) + \zeta_2\zeta_3(\lambda_2 s'(\lambda_2)s(\lambda_3) - \lambda_3 s(\lambda_2)s'(\lambda_3))] \quad (33)$$

For preferred direction a parallel to e_1 (in the direction of fiber) and preferred direction a perpendicular to e_2 (perpendicular at the fiber direction). For equibiaxial test, $\lambda_1 = \lambda_2 = \lambda$ and $\lambda_3 = \frac{1}{\lambda_1\lambda_2} = \frac{1}{\lambda^2}$.

Therefore we obtain

$$\sigma_{11} = \lambda f'(\lambda)\left[\mu_T + 2(\mu_L - \mu_T)\zeta_1 + \frac{\beta}{2}\zeta_1^2\right] - \left(\frac{1}{\lambda^2}\right)f'\left(\frac{1}{\lambda^2}\right)\left[\mu_T + 2(\mu_L - \mu_T)\zeta_3 + \frac{\beta}{2}\zeta_3^2\right] + \beta\left[\begin{array}{l}\zeta_1\zeta_2\lambda s'(\lambda)s(\lambda) + \zeta_1\zeta_3\left(\lambda s'(\lambda)s\left(\frac{1}{\lambda^2}\right) - \left(\frac{1}{\lambda^2}\right)s(\lambda)s'\left(\frac{1}{\lambda^2}\right)\right) - \\ \zeta_2\zeta_3\left(\frac{1}{\lambda^2}\right)s(\lambda)s'\left(\frac{1}{\lambda^2}\right)\end{array}\right] \quad (34)$$

Since $\zeta_1 = a \cdot e_1$, $\zeta_2 = a \cdot e_2$, and $\zeta_3 = a \cdot e_3$, where $a // e_1$, $a \perp e_2$, $a \perp e_3$, then $\zeta_1 = 1$, $\zeta_2 = 0$ and $\zeta_3 = 0$

$$\sigma_{11} = \lambda f'(\lambda)\left[\mu_T + 2(\mu_L - \mu_T) + \frac{\beta}{2}\right] - \left(\frac{1}{\lambda^2}\right)f'\left(\frac{1}{\lambda^2}\right)\mu_T$$

$$\sigma_{11} = \mu_T\left[\lambda f'(\lambda) - \left(\frac{1}{\lambda^2}\right)f'\left(\frac{1}{\lambda^2}\right)\right] - \lambda f'(\lambda)\left[2(\mu_L - \mu_T) + \frac{\beta}{2}\right] \quad (35)$$

Penerbit Akademia Baru
Journal of Advanced Research in Applied Mechanics
ISSN (online): 2289-7895 | Vol. 5, No.1. Pages 15-29, 2015

is parallel to the fiber direction. Similarly for σ_{22}

$$\sigma_{22} = \lambda f'(\lambda)\left[\mu_T + 2(\mu_L - \mu_T)\zeta_2 + \frac{\beta}{2}\zeta_2^2\right] - \left(\frac{1}{\lambda^2}\right) f'\left(\frac{1}{\lambda^2}\right)\left[\mu_T + 2(\mu_L - \mu_T)\zeta_3 + \frac{\beta}{2}\zeta_3^2\right] + \beta\left[\begin{array}{l}\zeta_1\zeta_2\lambda s(\lambda)s'(\lambda) - \zeta_1\zeta_3\left(\frac{1}{\lambda^2}\right)\lambda_3 s(\lambda_1)s'\left(\frac{1}{\lambda^2}\right) + \\ \zeta_2\zeta_3\left(\lambda s'(\lambda)s\left(\frac{1}{\lambda^2}\right) - \left(\frac{1}{\lambda^2}\right)s(\lambda)s'\left(\frac{1}{\lambda^2}\right)\right)\end{array}\right]$$

$$\begin{aligned}\sigma_{22} &= \lambda f'(\lambda)\mu_T - \left(\frac{1}{\lambda^2}\right) f'\left(\frac{1}{\lambda^2}\right)\mu_T \\ &= \mu_T\left[\lambda f'(\lambda) - \left(\frac{1}{\lambda^2}\right) f'\left(\frac{1}{\lambda^2}\right)\right]\end{aligned} \quad , \tag{36}$$

is perpendicular at the fiber direction. Therefore we obtain:

$$\begin{aligned}\sigma_{11} - \sigma_{22} &= \lambda f'(\lambda)\left[2(\mu_L - \mu_T) + \frac{\beta}{2}\right] \\ \sigma_{11} &= \sigma_{22} + \lambda f'(\lambda)\left[2(\mu_L - \mu_T) + \frac{\beta}{2}\right]\end{aligned} \tag{37}$$

2.3 Application to Mitral Valve Leaflet

The advantage of this model, in a triaxial test of an incompressible solid, where $W_e = W(\lambda_1, \lambda_2, \zeta_1, \zeta_2)$, the principal stretches λ_1 and λ_2 can be provided independently. The invariants ζ_1 and ζ_2 can be varied independently by taking different samples, of the same material, with different preferred directions (relative to a principal direction, a is a preferred direction, σ_{11} is parallel to fiber and σ_{22} is perpendicular to fiber).

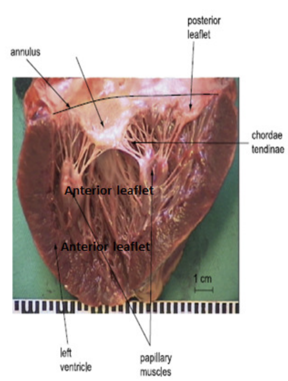

Figure 3: Mitral apparatus of human heart source [19]

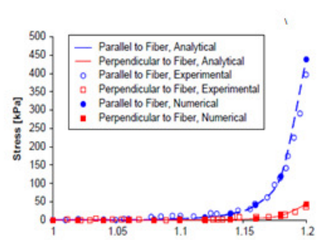

Figure 4: Equibiaxial strain applied to anterior leaflet: source [8]

Journal of Advanced Research in Applied Mechanics
ISSN (online): 2289-7895 | Vol. 5, No.1. Pages 15-29, 2015

In previous work [20,21] Maple and standard least square method has been used to fit the theoretical curve to experimental data [6]. In this paper we use Maple 13 and Mathematica 9,0 to determined accurate material constants after make comparison to the result of both methods.

Table 1 : Typical equibiaxial stress-stretch data σ_{11}, σ_{22} and $\sigma_{11}-\sigma_{22}$ of anterior mitral valve leaflet.

λ	1.029	1.062	1.070	1.1329	1.1370	1.152	1.164	1.173	1.1809	1.186	1.192	1.200
σ_{11}	0.000	0.000	5.754	20.934	19.311	27.983	48.721	62.819	112.700	143.270	229.270	401.679
σ_{22}	0.000	0.000	4.131	5.754	5.754	12.262	13.484	10.230	18.902	17.820	24.869	38.418
$\sigma_{11}-\sigma_{22}$	0.000	0.000	1.623	15.180	13.557	15.557	35.237	52.590	93.795	125.246	204.401	363.261

Two function such as $\lambda f'(\lambda)=\lambda(\lambda-1)^{\alpha}e^{\beta x}$ for σ_{11} (parallel to the fiber) and $\lambda f'(\lambda)=\lambda^{\alpha}(\lambda-1)$ for σ_{22} (perpendicular to the fiber) are used to investigate and substitute into the biaxial constitutive equation. These two functions represent the ground state of the curve. We used Least squares Method by MAPLE 13 and Mathematica 9.0 in curve fitting to determine the material constants.

3.0 RESULTS AND DISCUSSION

3.1 Curve Fitting Methods

Case 1: For the stretch deformation parallel to fiber (equation 35), the graph is expected to be an exponential curve and the polinomial ,we proposed the constitutive equation of the form $\lambda f'(\lambda)=\lambda(\lambda-1)^{\alpha}e^{\gamma\lambda}$ gives the results as follow:

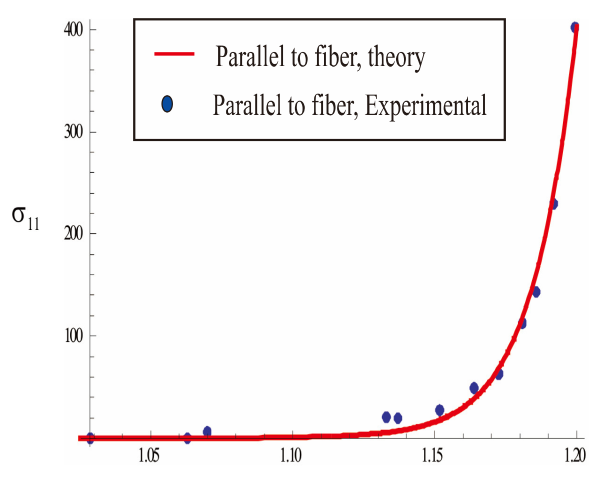

Figure 5: The graph of the curve fitting in the parallel direction to the fiber

Case 2: For the stretch deformation pependicular to fiber (equation 36) ,we proposed the constitutive equation of the form $\lambda f'(\lambda) = \lambda^{\alpha}(\lambda - 1)$ gives the result as follow:

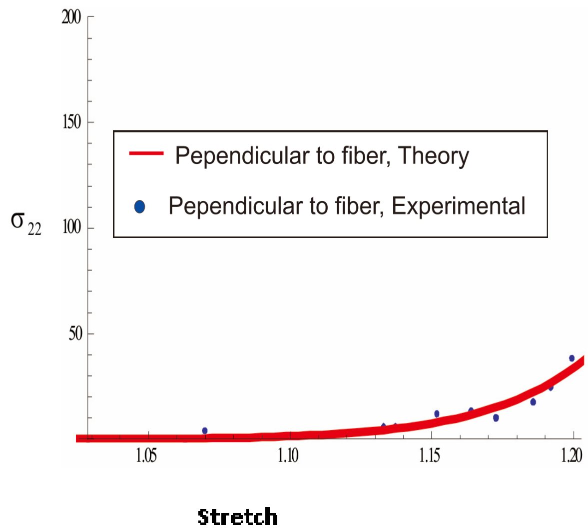

Figure 6: The graph of curve fitting in the direction perpendicular to the fiber

Case 3: For $\sigma_{11} - \sigma_{22}$ (equation 37), the graph is an exponential curve and the polinomial ,we proposed the constitutive equation of the form $\lambda f'(\lambda) = \lambda(\lambda - 1)^{\alpha} e^{\gamma\lambda}$ gives the results as follow:

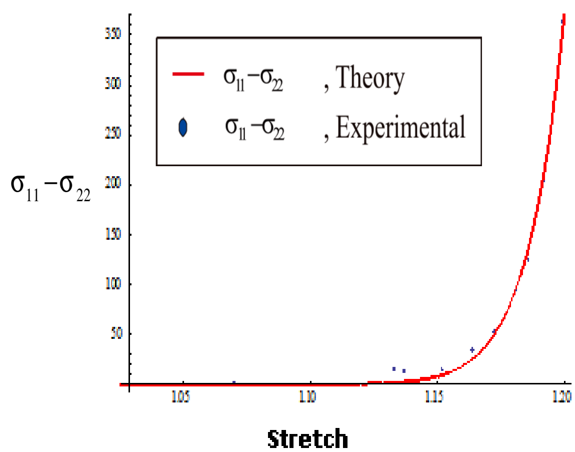

Figure 7: The graph the curve fitting for $\sigma_{11} - \sigma_{22}$

3.1 Elastic Contants

Since the Maple curve fitting method have limitation on the form of constitutive equation where is not all the parameters can be determined with the unique values, and the only value for the stretch deformation perpendicular to the fiber we obtain μ_T = 0.9988 see table 2.

Furthermore α and γ have to determine with values 9 and 18 respectively. Using Mathematica curve fitting method is better, where all the value of parameters can be determined in a single process.

Table 2: Elastic constants obtain from the curve fitting process by MAPLE 13

Constitutive Model	stress	α	γ	Equation in term of μ_T, μ_L, β
$\lambda f'(\lambda) = \lambda(\lambda-1)^{\alpha} e^{\gamma\lambda}$	σ_{11}	9	18	$2(\mu_L - \mu_T) + \frac{\beta}{2} = 0.2643$, $\mu_T = 2.3848$
	$\sigma_{11} - \sigma_{22}$	10	18	$2(\mu_L - \mu_T) + \frac{\beta}{2} = 1.2572$
$\lambda f'(\lambda) = (\lambda-1)\lambda^{\alpha}$	σ_{22}	28		$\mu_T = 0.9988$

Table 2: Material constants obtain from the curve fitting process by Mathematica 9.0

Constitutive Model	stress	α	γ	μ_T	μ_L	β
$\lambda f'(\lambda) = \lambda(\lambda-1)^{\alpha} e^{\gamma\lambda}$	σ_{11}	9	18	2.3848	0.6623	2.6491
	$\sigma_{11} - \sigma_{22}$	10	18	0.6545	1.3455	2.3821
$\lambda f'(\lambda) = (\lambda-1)\lambda^{\alpha}$	σ_{22}	28		0.9988		

4.0 CONCLUSION

From all the graphs of the curve fitting figure 5, figure 6 and figure 7 shows that the theory compares well to the experimental data. Elastic constants will be determined by make a comparison to the result from table 2 and table 3. It's clearly that μ_T = 0.9988 are identically from both tables. With the condition that the elastic constants must be unique and μ_L, μ_T and β must be greater than zero where $\mu_L > \mu_T$. Therefore the value of the elastic constants are μ_L = 1.3455 and β=2.3821. We only use μ_L, μ_T and β to predict the experiment and with the simple form of constitutive equation in-terms of physical invariants has an advantage to carry out experiments. In the near future, this constitutive model will be compared with various types of experiment data and with type of materials.

REFERENCES

[1] M.J. Fagan, Finite Element Analysis, Theory and Practice, Longman Group UK Limited, Malaysia (1992).

[2] G.A. Holzapfel, Nonlinear Solid Mechanics: A Continuum Approach for Engineering, Wiley (2001).

[3] P. Tong, Y.C. Fung, The stress-strain relationship for the skin, Journal of Biomechanics 7 (1974) 649-657.

[4] R.S. Rivlin, The Valanis-Landel Strain Energy Function, Journal of Elasticity 73 (2003) 291-297.

[5] G. Marckmann, E. Verron, Comparison of Hyperelastic Models for Rubber-Like Materials, Rubber Chemistry and Technology 79 (2006) 835-858.

[6] C. Chui, E. Kobayashi, X. Chen, T. Hisada, I. Sakuma, Transversely isotropic properties of porcine liver tissue: Experiments and constitutive modeling, Medical & Biological Engineering & Computing 45 (2007) 99-106.

[7] K. May-Newman, F.C.P. Yin, Biaxial mechanical behavior of excised porcine mitral valve leaflets, American Physiological Society 269 (1995) H1319-H1327.

[8] K. May-Newman, F.C.P. Yin, A constitutive law for mitral valve tissue, Journal of Biomechanical Engineering 120 (1998) 38-46.

[9] Y.C. Fung, Elasticity of Soft Tissues in Simple Elongation, American Journal of Physiology 28 (1967) 1532-1544.

[10] A.J.M. Spencer, Deformations of Fibre-Reinforced Materials, Oxford University Press (1972).

[11] J.C. Criscione, A.S. Douglas, W.C. Hunter, Physically based invariants set for materials exhibiting transversely isotropic behaviour, Journal of the Mechanics and Physics of Solids 49 (2001) 871-891.

[12] J. Lu, L. Zhang, Physically motivated invariant formulation for transversely isotropic hyperelasticity, International Journal of Solids and Structures 42 (2005) 6015-6031.

[13] G. deBotton, L. Hariton, E.A. Socolsky, Neo-hookean fiber-reinforced composites in finite elasticity, Journal of the Mechanics and Physics of Solids 54 (2006) 533-559.

[14] Z.Y. Guo, X.Q. Peng, B.A. Moran, Composites-based hyperelastic constitutive model for soft tissue with application to the human annulus fibroses, Journal of the Mechanics and Physics of Solids 54 (2006) 1952-1971.

[15] M.B. Rubin, M. Jabareen, Physically based invariants for nonlinear elastic orthotropic solids, Journal of Elasticity 90 (2008) 1-18.

[16] A.J.M. Spencer, Constitutive theory for strongly anisotropic solids, continuum theory of the mechanics of fibre-reinforced composites, Springer Verlag, 282 (1984) 1-32.

[17] M.H.B.M. Shariff, Nonlinear transversely isotropic solid: An alternative representation, Quarterly Journal of Mechanics and Applied Mathematics 61 (2008) 129-149.

[18] M.H.B.M Shariff, B.A. Mahad , Zainal Abdul Aziz, On the Correlation of Theory and Experiment for Transversely Isotropic Nonlinear Incompressible Solids, Proceedings of the 9th International Conference on Computational Structures Technology, Athens, Greece (2008).

[19] V.Prot, Modelling and numerical analysis of the porcine and human mitral apparatus, Ph.D, Department of Structural Engineering, NTNU, Norway (2008)

[20] B.A. Mahad, M.H.B.M Shariff, Mukheta Isa, Zainal Abdul Aziz, Physical Interpretation Constitutive Model for Biological Tissues with Application to Mitral Valve Leaflet Tissues, Proceedings of the International Conference on Applied Analysis and Mathematical Modeling (ICAAMM2013), Istanbul, Turkey (2013).

[21] B.A. Mahad, M.H.B.M Shariff, Mukheta Isa, Zainal Abdul Aziz, Constitutive Equations of Mitral Valve Leaflet in Terms of Principal Stretches, Proceedings of Global Academic Network International Dubai Conference, Dubai (2014)